Local set approximation: infinitesimal to local theorems and applications

Stephen Lewis

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2014

Reading Committee:

Tatiana Toro, Chair

John Lee

Boris Solomyak

Program Authorized to Offer Degree:
Department of Mathematics

University of Washington

Abstract

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Stephen Lewis

Chair of the Supervisory Committee:
Professor Tatiana Toro
Mathematics

In this thesis we develop the theory of Local Set Approximation (LSA), a framework which arises naturally from the study of sets with singularities. That is, we describe the local structure of a set A in Euclidean space through studying a class of sets $\mathcal{S}$ which approximates A well in small balls. We will give two interpretations LSA in Chapters 2 and 3. If in small balls $B(x,r)$, our set A is close to some $S_{x,r} \in \mathcal{S}$, the approximation is *unilateral*. On the other hand, if in small balls $B(x,r)$, our set A is close to $S_{x,r}$ *and* $S_{x,r}$ is close to A, the approximation is *bilateral*. Both of these models appear naturally in areas of geometric measure theory such as area minimizers, mass minimizers, free boundaries, and regularity of measures. In Chapters 4 and 5, we give applications of local set approximation to the study of asymptotically optimally doubling measures.

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ACKNOWLEDGMENTS

In my first year at UW, I took my first course in Real Analysis with Tatiana Toro. After a discussion about the Kakeya problem following class, I knew that I wanted to study Geometric Measure Theory with Tatiana as my advisor. In the four years to follow, her guidance has been kind and plentiful. The varied set of questions I've had the opportunity to study has originated from her, and she always found ways to help me pursue my interests.

I owe many thanks to Matthew Badger for our collaboration. His insights began the study of Local Set Approximation as a study unto itself—a study which has captivated me at every step.

My time at the University of Washington has been filled with a very rich personal community. It would be impossible to thank the many people who made my time at UW one of intense personal growth. However, I owe a very dear thanks to Andrew Laszlo for our five inseparable years. Starting and finishing graduate school together has brought good into every piece of my life.

The research was partially supported by the National Science Foundation grant number DMS 0838212.

DEDICATION

To my family.

Everything I do
started and continues to grow
from you.

Chapter 1

LOCAL SET APPROXIMATION ARISES FROM PLATEAU'S PROBLEM

Dating back to Euler and Lagrange, soap-films and soap-bubbles have fascinated mathematicians and physicists. Roughly speaking, a soap-film is a stable configuration made by dipping a wire frame into a mixture of soap into water. A soap-bubble configuration is a stable configuration of bubbles made from such a mixture, without a fixed boundary like a wire frame. In the 1880's, the Belgian physicist Joseph Plateau described the structure of soap-films and soap-bubble configurations through experimental observation. His observations can be summarized in the following three laws, known as Plateau's Laws:

i. Soap films and soap bubbles are composed of pieces of entire smooth surfaces of constant mean curvature.

ii. These pieces meet in threes along smooth curves at $120°$ angles.

iii. These curves meet in isolated points points at $\arccos(-1/3) \approx 109.47°$ angles.

This physical study of soap-bubbles and soap-films leads to the question, *What mathematical model describes the behavior of soap-bubbles and soap-films?* A simple model (ignoring gravity, air currents, etc...) predicts that the soap film will try to minimize the energy due to tension, and thus stabilize at a local minimum of tension. Further, the tension will be proportional to area, and so we posit that local minima of the area functional will describe the structure of soap films.

In 1760 (long before Plateau's physical descriptions), the mathematician Lagrange asked the question; Given a boundary in space, is there an area minimizing surface with this boundary? Due to its connection with stable soap films, this has come to be known as Plateau's Problem, though it was stated more than a century before Plateau's study of soap films.

With the birth of Geometric Measure Theory in the 1930's, a new set of techniques and perspectives arrived which were very well suited to the Plateau Problem. Various solutions to Plateau's problem appeared, and we highlight a few. In the mid 1930's, Tibor Rado and Jessie Douglas independently proved existence of area minimizing surfaces with boundaries given by a simple closed curve. The solution of Rado requied the curve to be rectifiable, but the solution of Douglas is completely general. Both, however, yield solutions which are sometimes immersed, a property not well suited to the physical world. Moreover, their solutions do not include the types of singularities seen in soap films.

In 1960, E. R. Reifenberg provided a solution of Plateau's problem where he considered the class of sets with a given boundary A to be the sets containing A which are not "retractable" onto A. That is, surfaces which contain A and have a topological condition ensuring that they do not have "holes." Reifenberg proved that area minimizers in this class exist. Not only this, he proves that m-dimensional area minimizers in $\mathbb{R}^n$ exist for any given boundary A, the first existence theorem for m-minimizers in $\mathbb{R}^n$ without any restriction on n or m. Additionally, he proved that such minima are $\mathcal{H}^m$-almost everywhere locally Euclidean of dimension m. Key to his study was the quantification of *flatness* of a set in a small ball. This is called the *Reifenberg flatness* of a set, and it has become a central idea in many areas of GMT, Harmonic Analysis, and Minimal Surface Theory. Proving that minimal sets are almost everywhere Reifenberg flat is an example of using local set approximation to give a regularity statement. Reifenberg also showed higher regularity

While Reifenberg used sets and Hausdorff m-dimensional measure, other models were being developed with currents and varifolds. Roughly speaking, currents can be viewed as oriented weak duals to surfaces, and varifolds are their strictly positive counterparts. There were some solutions to Plateau's problem using currents and varifolds. However, these solutions assume the existence of a *boundary operator*, an operator which applied extra algebraic structure to the boundary. Almgren wished to study Plateau's problem for area-minimizers without the assumption of a boundary operator. As he wrote [15] "We do not wish to assume the existence of a boundary operator. . . because in many of the geometric, physical, and biological phenomena to which the results and methods are applicable there seems to be no natural notion of such a boundary operator." Almgren's paper establishes a solution to Plateau's problem using sets and the area functional.

However, despite several solutions to Plateau's Problem in its many modern formulations, it was not until the 1970's that significant progress was made in establishing Plateau's Laws by the mathematical models. In modern language, this could be called "establishing regularity." Jean Taylor studied the structure of soap bubbles and soap films by first classifying their infinitesimal behavior and then describing their local structure as smooth ($C^{1,\beta}$) deformations of this infinitesimal structure. Taylor proves that the tangent at any point

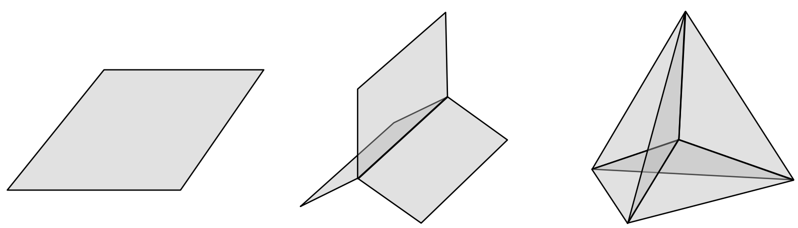

Figure 1.1: The infinitesimal structures of soap bubbles and soap films. From left to right: planes, Y-type sets, and T-type sets.

was one of three types of sets: The first is a 2-plane. The second is the union of three half spaces meeting along a line at 120° angles. We call these Y-type sets. The third can be described as the cone over the spine of a tetrahedron, or 6 "wedges" of space with angle $\theta_T = \arccos(-1/3)$ meeting edge to edge and corner to corner equally spaced about the origin. We call these T-type sets. Points whose tangent is a plane are the smooth points of Plateau's Laws. Points whose tangent is a Y-type set are the points in an edge of three surfaces meeting. Points whose tangent is a T-point are the points where four edges (and six surfaces) meet. Again, Taylor's work establishes regularity using variants of local set approximation.

Later, G. David established regularity theorems similar to Taylor's for 2-dimensional minimal sets in $\mathbb{R}^n$ [7, 8]. Following the same path, David first gives a partial classification of the possible infinitesimal behaviors into planes, Y-type sets, and T-type sets, where now T-type sets are to be interpreted loosely as sets with "high density" at the origin. In fact, it is known that the classification of Taylor does not extend blindly to $\mathbb{R}^n$: for example, X. Liang recently showed that two intersecting 2-planes in $\mathbb{R}^4$ which are almost orthogonal form a minimal surface. Despite not having a representative list minimal cones, David is able to show that at each point, there is some open neighborhood which is a deformation of some minimal cone. This is done by gaining control on the structure of the T-type sets and using a local approximation scheme.

In the regularity results of Reifenberg, Taylor, and David, the key idea is comparing a local minimum to a class of model sets. More precisely, Reifenberg compared small pieces of the minimum about a flat point to planes. Taylor uses comparison to the 2-dimensional minimal cones in $\mathbb{R}^3$ (planes, Y-type sets, and T-type sets). David uses the 2-dimensional minimal cones in $\mathbb{R}^n$, working with certain quantitative properties of the T-type sets in absence of a complete list. This type of argument could be characterized as using local set approximation to classify regularity.

In this thesis, we investigate local set approximation in its own right. We aim to provide a unifying framework to ideas which have existed in various areas of geometric measure theory and variational analysis (see Theorems 2.4.2 and 2.5.2), as well as extend the strength of the existing theory. In Chapter 2, we develop the theory of set convergence and local set approximation in depth. We will define two types of infinitesimal structures to a set A; its *tangent sets* and *pseudotangent sets*, and connect tangents and pseudotangets to approximability. This will allows us to analyze the "regular" and "singular" pieces of sets. In Chapter 3, we study unilateral local set approximation, a model which will be very well suited to study dimension bounds on singular sets. We develop a parallel theory of *upper tangents* and *upper pseudotangents* which we will use to provide Minkowski dimension bounds on the singular parts of sets. In Chapter 4, we study the local structure of the support of asymptotically optimally doubling measures using the theory developed in Chapters 2 and 3. In Chapter 5, we explore "high regularity" local set approximation through a case study of Hölder asymptotically doubling measures on $\mathbb{R}^4$.

Chapter 2
BILATERAL LOCAL SET APPROXIMATION

2.1 Introduction

In this chapter, we investigate the structure of sets $A \subseteq \mathbb{R}^n$ that admit uniform local approximations by a class of model sets $\mathcal{S}$. We consider sets A which have one of the following forms.

- A is *locally ε-approximable by $\mathcal{S}$* if for all compact sets $K \subseteq A$, there exists a scale r_K such that for all $x \in K$ and $0 < r \leq r_K$, there exists a set $S_{x,r} \in \mathcal{S}$, for which A is ε-close to $S_{x,r}$ near x at scale r.

- A is *locally well approximated by $\mathcal{S}$* if A is locally ε-approximable by $\mathcal{S}$ for all $\varepsilon > 0$.

Different meanings may be attached to the phrase "A is ε-close to $S_{x,r}$ near x at scale r", resulting in different models of local set approximation. We explore several possibilities below.

The principal distinction between models of local set approximation that have appeared in the literature is the directionality of approximation; that is, the symmetry or asymmetry of approximation measurements. On one hand, if distance between an approximated set A and an approximating set $S_{x,r}$ is measured by how close A is to $S_{x,r}$ *and* by how close $S_{x,r}$ is to A, then the approximation is *bilateral.* On the other hand, if distance between A and its approximant $S_{x,r}$ is measured only by how close A is to $S_{x,r}$, then the approximation is *unilateral.* The decision to use a bilateral or unilateral approximation model should depend on the application of the model.

Example. [$\mathcal{S} = \mathcal{G}(n, m)$, bilateral approximation: Reifenberg flat sets] The prototypical example of local set approximation is due to E. R. Reifenberg [30], who studied sets which admit uniform bidirectional local approximations by m-dimensional planes in $\mathbb{R}^n$, $1 \leq m \leq$

$n-1$, in the context of solving Plateau problems in arbitrary codimension. Following the work of Kenig and Toro [18], these sets are now called Reifenberg flat sets. In the Reifenberg model, the local error between an approximated set A and an approximating plane $S_{x,r}$ is measured in terms of the (normalized) Hausdorff distance between $A \cap B(x,r)$ and $(x+S_{x,r}) \cap B(x,r)$. The initial application of Reifenberg flat sets, established in [30], is *the topological disk theorem*: for every $1 \leq m \leq n-1$, there exists a constant $\delta_{n,m} > 0$ such that if A is locally $\delta_{n,m}$-close to $\mathcal{G}(n,m)$ (in the sense above), then A is locally homeomorphic to $\mathbb{R}^m$.

Local approximation schemes of Reifenberg type have been looked at by several authors in the context of geometric measure theory and analysis/PDE on domains with rough boundary. The first example of Reifenberg flatness in the context of PDE is due to Kenig and Toro [18] who studied free boundary regularity via local flatness. For Almgren minimal sets in $\mathbb{R}^3$, this was considered by G. David, T. DePauw and T. Toro [10]; for boundaries of domains that are close to Lipschitz graphs by J. Lewis and K. Nyström [22]; for free boundaries in a two-phase regularity problem for harmonic measure, by M. Badger [2]; and for the supports of asymptotically optimally doubling measures by S. Lewis [23]. A common element in each of these Reifenberg type approximation schemes is that local errors between a set A and approximating sets $S \in \mathcal{S}$ are bilateral, measured using normalized Hausdorff distance between $A \cap B(x,r)$ and $(S+x) \cap B(x,r)$.

Example. [$\mathcal{S} = \mathcal{G}(n,m)$, unilateral linear approximation] For a compact set $K \subseteq \mathbb{R}^2$ and a cube Q, Jones [16] introduced the *Jones beta number* $\beta_\infty(x,r)$ defined by the infimum over all lines L which intersect $B(x,r)$ of the normalized unilateral Hausdorff distance from $B(x,r) \cap K$ and L. Jones defines $\beta(K)$ as a square sum over a geometrically decreasing mesh of cubes. In [16], Jones proves his celebrated Traveling Salesman Theorem, which says that for every set K, $\beta(K) < \infty$ if and only if K is contained in a connected, rectifiable set Γ. We remark that the Traveling Salesman Theorem in this context is a continuous analogue of the Traveling Salesman Problem as the term is used in graph theory and computer science.

Mattila and Vuorinen [26] introduced a similar unidirectional local approximation scheme, in connection with their study of the dimension of quasiconformal spheres. They say that a set $A \subset \mathbb{R}^n$ has the *m-dimensional linear approximation property* if it admits uniform unidirectional local approximations by m-dimensional planes in $\mathbb{R}^n$, $1 \leq m \leq n-1$. That is, at small enough scales in a locally uniform sense, $A \cap B(x,r)$ is always contained in the εr neighborhood around some plane P. Mattila and Vuorinen proved that sets which are unidirectionally locally ε-close to $\mathcal{G}(n,m)$ (in the sense above) have upper Minkowski dimension at most $m + C_{n,m}\varepsilon^2$ for some constant $C_{n,m} > 1$.

In [13], David and Toro study sets $K \subseteq \mathbb{R}^n$ with the m-dimensional linear approximation property. They show that there exist K with the m-dimensional linear approximation property for which there exists no set Γ containing K which is Reifenberg flat. The authors show, however, that by adding an additional compatibility condition on the approximating planes, that K can be extended to a Reifenberg flat set Γ.

The goal of this and the next chapter is to initiate the study of Reifenberg and Mattila-Vuorinen models of local set approximation in fuller generality than has been previously considered.

2.2 Distances between sets and convergence of closed sets

For any set $A \subseteq \mathbb{R}^n$, we let $\overline{A}$ denote the closure of A. For all $x \in \mathbb{R}^n$ and $r > 0$, we let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| \leq r\}$ denote the *closed* ball in $\mathbb{R}^n$ with center x and radius r. We also let $\mathfrak{C}(x)$ denote the collection of all closed sets in $\mathbb{R}^n$ containing x.

The basic building block that we use to construct distances between sets is the excess of one set over another. Let $A, B \subseteq \mathbb{R}^n$ be nonempty. The *excess of A over B* is the quantity defined by

$$\mathrm{e}(A,B) = \sup_{a \in A} \inf_{b \in B} |a-b| \in [0,\infty]. \tag{2.1}$$

Geometrically the excess of A over B is less than (at most) ε precisely when A is contained in the open (closed) ε-tubular neighborhood of B. Some key properties of excess include

- *closure*: $\mathrm{e}(A,B) = \mathrm{e}(A,\overline{B}) = \mathrm{e}(\overline{A},\overline{B}) = \mathrm{e}(\overline{A},B)$;

- *containment*: $\mathrm{e}(A,B) = 0$ if and only if $A \subseteq \overline{B}$;

- *monotonicity*: if $A \subseteq A'$ and $B \supseteq B'$, then $\mathrm{e}(A,B) \le \mathrm{e}(A',B')$; and

- *triangle inequality*: $\mathrm{e}(A,C) \le \mathrm{e}(A,B) + \mathrm{e}(B,C)$.

We emphasize that excess is an asymmetric quantity and is allowed to be infinite.

For all $x \in \mathbb{R}^n$ and $r > 0$ and for all $A \subseteq \mathbb{R}^n$ containing x, we define the *relative excess of A in $B(x,r)$ over B* as

$$\widetilde{\mathrm{d}}^{\,x,r}(A,B) := \frac{1}{r}\,\mathrm{e}(A \cap B(x,r), B) \in [0,\infty). \tag{2.2}$$

We include the factor $1/r$ in the definition of relative excess so that $\widetilde{\mathrm{d}}^{\,x,r}$ is scale invariant, in the sense that

$$\widetilde{\mathrm{d}}^{\,x,r}(A,B) = \widetilde{\mathrm{d}}^{\,\lambda x,\lambda r}(\lambda A, \lambda B) \quad \text{for all } \lambda > 0. \tag{2.3}$$

In contrast to the excess, the relative excess of one set over another (whenever defined) is always finite. Furthermore, if A and B both contain x, then $\widetilde{\mathrm{d}}^{\,x,r}(A,B) \le 1$. Relative excess inherits the following additional properties from excess.

Lemma 2.2.1. *Let $A, B, C \subseteq \mathbb{R}^n$, $x, y \in \mathbb{R}^n$ and $r, s > 0$.*

- closure*: If $x \in A$, then $\widetilde{\mathrm{d}}^{\,x,r}(A,B) = \widetilde{\mathrm{d}}^{\,x,r}(A,\overline{B}) \le \widetilde{\mathrm{d}}^{\,x,r}(\overline{A},\overline{B}) = \widetilde{\mathrm{d}}^{\,x,r}(\overline{A},B)$.*

- containment*: If $x \in A$, then $\widetilde{\mathrm{d}}^{\,x,r}(A,B) = 0$ if and only if $A \cap B(x,r) \subseteq \overline{B}$.*

- monotonicity*: If $B(x,r) \subseteq B(y,s)$, $x \in A$, $y \in A'$, $A \subseteq A'$ and $B \supseteq B'$, then*

$$\widetilde{\mathrm{d}}^{\,x,r}(A,B) \le \frac{s}{r}\,\widetilde{\mathrm{d}}^{\,y,s}(A',B'). \tag{2.4}$$

- strong quasitriangle inequality*: If $x \in A \cap \overline{B}$ and $\widetilde{\mathrm{d}}^{\,x,r}(A,B) \le \varepsilon$, then*

$$\widetilde{\mathrm{d}}^{\,x,r}(A,C) \le \widetilde{\mathrm{d}}^{\,x,r}(A,B) + (1+\varepsilon)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon)}(\overline{B},C). \tag{2.5}$$

- quasitriangle inequality*: If $x \in A \cap \overline{B}$, then*

$$\widetilde{\mathrm{d}}^{\,x,r}(A,C) \le \widetilde{\mathrm{d}}^{\,x,r}(A,B) + 2\,\widetilde{\mathrm{d}}^{\,x,2r}(\overline{B},C). \tag{2.6}$$

Proof. If $x \in A$, then

$$\mathrm{e}(A \cap B(x,r), B) = \mathrm{e}(A \cap B(x,r), \overline{B}) \leq \mathrm{e}(\overline{A} \cap B(x,r), \overline{B}) = \mathrm{e}(\overline{A} \cap B(x,r), B)$$

by the closure and monotonicity properties of excess. Dividing through the previous line by r establishes the closure property of relative excess. Likewise the containment property of relative excess follows immediately from the containment property of excess.

If $B(x,r) \subseteq B(y,s)$, $x \in A$, $y \in A'$, $A \subseteq A'$ and $B \supseteq B'$, then

$$\widetilde{\mathrm{d}}^{\,x,r}(A,B) = \frac{1}{r}\,\mathrm{e}(A \cap B(y,s), B) \leq \frac{1}{r}\,\mathrm{e}(A' \cap B(y,s), B') = \frac{s}{r}\,\widetilde{\mathrm{d}}^{\,y,s}(A', B')$$

by monotonicity of excess. This establishes the monotonicity property of relative excess.

Suppose that $x \in A \cap \overline{B}$ and $\widetilde{\mathrm{d}}^{\,x,r}(A,B) \leq \varepsilon$ for some $\varepsilon > 0$. Fix $a \in A \cap B(x,r)$ and fix $\delta > 0$. By the closure and monotonicity properties of excess, there is $\bar{b} \in \overline{B}$ such that

$$\left|a - \bar{b}\right| = \mathrm{e}(\{a\}, B) \leq \mathrm{e}(A \cap B(x,r), B) = r\,\widetilde{\mathrm{d}}^{\,x,r}(A,B) \leq r\varepsilon.$$

Similarly, there is $c \in C$ such that

$$\left|\bar{b} - c\right| \leq \mathrm{e}(\{\bar{b}\}, C) + \delta \leq \mathrm{e}(\overline{B} \cap B(x, r(1+\varepsilon)), C) + \delta = r(1+\varepsilon)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon)}(\overline{B}, C) + \delta.$$

Hence, combining the two displayed equations,

$$\mathrm{e}(\{a\}, C) \leq \left|a - \bar{b}\right| + \left|\bar{b} - c\right| \leq r\,\widetilde{\mathrm{d}}^{\,x,r}(A,B) + r(1+\varepsilon)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon)}(\overline{B}, C) + \delta$$

Letting $\delta \to 0$ and then taking the supremum over all a in $A \cap B(x,r)$, we obtain

$$\mathrm{e}(A \cap B(x,r), C) \leq r\,\widetilde{\mathrm{d}}^{\,x,r}(A,B) + r(1+\varepsilon)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon)}(\overline{B}, C).$$

The strong quasitriangle inequality follows by dividing the last line through by r.

If $x \in A \cap B$, then $\widetilde{\mathrm{d}}^{\,x,r}(A,B) = \widetilde{\mathrm{d}}^{\,x,r}(A,\overline{B}) \leq 1$. Thus, the quasitriangle inequality follows from the strong quasitriangle inequality. □

We now define the relative Walkup-Wets distance. First however, we note that similar quantities are often named by some variant of Hausdorff distance. However, Hausdorff's contribution to the notion of "distance" between sets was to take the sum of two quantities where below we will take a maximum. The definition with a maximum had been used before Hausdorff, and with a quantity more similar to ours, by Walkup and Wets. See [31] for a more detailed history. For all $x \in \mathbb{R}^n$ and $r > 0$, the *relative Walkup-Wets distance in* $B(x,r)$ is given by

$$\widetilde{\mathrm{D}}^{x,r}[A,B] := \max\left\{\widetilde{\mathrm{d}}^{\,x,r}(A,B), \widetilde{\mathrm{d}}^{\,x,r}(B,A)\right\} \in [0,1] \tag{2.7}$$

for all $A, B \subseteq \mathbb{R}^n$ such that $x \in A \cap B$. The relative Walkup-Wets distance satisfies the following properties.

Lemma 2.2.2. *Let* $A, B, C \subseteq \mathbb{R}^n$*,* $x, y \in \mathbb{R}^n$ *and* $r, s > 0$*. Assume that* $x, y \in A \cap B \cap C$*.*

- closure*:* $\widetilde{\mathrm{D}}^{x,r}[A,B] \leq \widetilde{\mathrm{D}}^{x,r}[\overline{A},\overline{B}]$.

- containment*:* $\widetilde{\mathrm{D}}^{x,r}[A,B] = 0$ *if and only if* $A \cap B(x,r) \subseteq \overline{B}$ *and* $B \cap B(x,r) \subseteq \overline{A}$*. In particular,* $\widetilde{\mathrm{D}}^{x,r}[\overline{A},\overline{B}] = 0$ *if and only if* $\overline{A} \cap B(x,r) = \overline{B} \cap B(x,r)$*.*

- monotonicity*: If* $B(x,r) \subseteq B(y,s)$*, then*

$$\widetilde{\mathrm{D}}^{x,r}[A,B] \leq \frac{s}{r}\,\widetilde{\mathrm{D}}^{y,s}[A,B] \tag{2.8}$$

- strong quasitriangle inequality*: If* $\widetilde{\mathrm{D}}^{x,r}[A,B] \leq \varepsilon_1$ *and* $\widetilde{\mathrm{D}}^{x,r}[B,C] \leq \varepsilon_2$*, then*

$$\widetilde{\mathrm{D}}^{x,r}[A,C] \leq (1+\varepsilon_2)\,\widetilde{\mathrm{D}}^{x,r(1+\varepsilon_2)}[A,\overline{B}] + (1+\varepsilon_1)\,\widetilde{\mathrm{D}}^{x,r(1+\varepsilon_1)}[\overline{B},C], \tag{2.9}$$

- quasitriangle inequality*:*

$$\widetilde{\mathrm{D}}^{x,r}[A,C] \leq 2\,\widetilde{\mathrm{D}}^{x,2r}[A,\overline{B}] + 2\,\widetilde{\mathrm{D}}^{x,2r}[\overline{B},C]. \tag{2.10}$$

- scale-invariance*:*

$$\widetilde{\mathrm{D}}^{x,r}[A,B] = \widetilde{\mathrm{D}}^{\lambda x,\lambda r}[\lambda A,\lambda B] \quad \text{for all } \lambda > 0. \tag{2.11}$$

Proof. Suppose that $\widetilde{\mathrm{D}}^{x,r}[A,B] \leq \varepsilon_1, \widetilde{\mathrm{D}}^{x,r}[B,C] \leq \varepsilon_2$. since $\widetilde{\mathrm{d}}^{\,x,r}(A,B) \leq \varepsilon_1$,

$$\begin{aligned}\widetilde{\mathrm{d}}^{\,x,r}(A,C) &\leq \widetilde{\mathrm{d}}^{\,x,r}(A,\overline{B}) + (1+\varepsilon_1)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_1)}(\overline{B},C)\\ &\leq (1+\varepsilon_2)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_2)}(A,\overline{B}) + (1+\varepsilon_1)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_1)}(\overline{B},C)\end{aligned}$$

by the strong quasitriangle inequality and monotonicity. Similarly, since $\widetilde{\mathrm{d}}^{\,x,r}(C,B) \le \varepsilon_2$,

$$\widetilde{\mathrm{d}}^{\,x,r}(C,A) \le (1+\varepsilon_1)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_1)}(C,\overline{B}) + (1+\varepsilon_2)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_2)}(\overline{B},A).$$

Thus, noting that $\max\{a+b, c+d\} \le \max\{a,d\} + \max\{b,c\}$, we have

$$\begin{aligned}
\widetilde{\mathrm{D}}^{x,r}[A,C] =& \max\{\widetilde{\mathrm{d}}^{\,x,r}(A,C), \widetilde{\mathrm{d}}^{\,x,r}(C,A)\} \\
\le& \max\{(1+\varepsilon_2)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_2)}(A,\overline{B}) + (1+\varepsilon_1)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_1)}(\overline{B},C), \\
&\qquad (1+\varepsilon_1)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_1)}(C,\overline{B}) + (1+\varepsilon_2)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_2)}(\overline{B},A)\} \\
\le& \max\{(1+\varepsilon_2)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_2)}(A,\overline{B}), (1+\varepsilon_2)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_2)}(\overline{B},A)\} \\
&\quad + \max\{(1+\varepsilon_1)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_1)}(\overline{B},C), (1+\varepsilon_1)\,\widetilde{\mathrm{d}}^{\,x,r(1+\varepsilon_1)}(C,\overline{B})\} \\
=& (1+\varepsilon_2)\,\widetilde{\mathrm{D}}^{x,r(1+\varepsilon_2)}[A,\overline{B}] + (1+\varepsilon_1)\,\widetilde{\mathrm{D}}^{x,r(1+\varepsilon_1)}[\overline{B},C].
\end{aligned}$$

Thus, we have established the strong quasitriangle inequality. The quasitriangle inequality follows from the strong quasitriangle inequality, because $\widetilde{\mathrm{D}}^{x,r}[A,B] \le 1$ and $\widetilde{\mathrm{D}}^{x,r}[B,C] \le 1$. □

Definition 2.2.3. *Let $A, A_1, A_2, \cdots \in \mathfrak{C}(x)$. We say that A_i* converges *to A* in $\mathfrak{C}(x)$ *if $\widetilde{\mathrm{D}}^{x,r}[A,A_i] \to 0$ as $i \to \infty$ for all $r > 0$.*

Theorem 2.2.4. *The space $\mathfrak{C}(x)$ is sequentially compact. That is, for every sequence (S_i) in $\mathfrak{C}(x)$, there is a subsequence (S_{ij}) of (S_i) and a set $S \in \mathfrak{C}(x)$ such that $S_{ij} \to S$ in $\mathfrak{C}(x)$.*

Proof. E.g., see Theorem 4.18 in [31]. □

Remark 2.2.5. *It is an unfortunate fact that the relative Walkup-Wets distance $\widetilde{\mathrm{D}}^{x,r}[A,B]$ does not satisfy the triangle inequality. However, the proofs to follow demonstrate the power of the quasitriangle inequality and strong quasitriangle inequality. To obtain a triangle inequality, one might be tempted to instead define the relative Hausdorff distance $\mathrm{D}^{x,r}[A,B]$,*

$$\mathrm{D}^{x,r}[A,B] = \frac{1}{r}\max\left\{\mathrm{e}(A\cap B(x,r), B\cap B(x,r)), \mathrm{e}(B\cap B(x,r), A\cap B(x,r))\right\}, \qquad (2.12)$$

and declare that $A_i \to A$ in $\mathfrak{C}(x)$ if $\mathrm{D}^{x,r}[A_i,A] \to 0$ as $i \to \infty$ for all $r > 0$. Although the relative Hausdorff distance does satisfy the triangle inequality, it is deficient in other important respects. Most importantly, $\mathfrak{C}(x)$ is not sequentially compact with respect to this *definition of convergence. For example, the sequence* $A_i = \{0, 1 + 1/i\} \subseteq \mathbb{R}$ *does not converge to* $A = \{0, 1\}$ *because* $\mathrm{D}^{0,1}[A_i, A] = 1$ *for all* $i \geq 1$*. In fact, neither* A_i *nor any of its subsequences converge to any set in* $\mathfrak{C}(x)$ *using the relative Hausdorff distance definition. A related difficulty of the relative Hausdorff distance is that it does not satisfy the monotonicity property.*

However, it should be noted that in the case when A *is a cone with vertex at* x, $\widetilde{\mathrm{D}}^{x,r}[A, B] \leq \mathrm{D}^{x,r}[A, B] \leq 2\, \widetilde{\mathrm{D}}^{x,r}[A, B]$*, and so the quantities are equivalent. Also in this case,* $\widetilde{\mathrm{D}}^{x,r}[A, B]$ *will satisfy a near-monotonicity property. See Lemma A.1.2 and Lemma A.1.3.*

Local set approximation has most often been applied in the case of planar approximation, and of course it is always true that a plane containing x *is a cone with vertex at* x*. In this sense, local approximability with planes may be considered with either the Walkup-Wets distance or the relative Hausdorff distance. Much interesting work has been done with Reifenberg flatness using the relative Hausdorff distance (see Definition 2.3.1 and previous examples). See the discussion in the introduction of this chapter for a more complete reference.*

Lemma 2.2.6. *Let* $A, A_i, B \in \mathfrak{C}(x)$*, and assume that* $A_i \to A$ *in* $\mathfrak{C}(x)$*. Then for any* $r > 0$ *and* $\varepsilon \in (0, 1)$*,*

$$(1-\varepsilon) \limsup_{i\to\infty} \widetilde{\mathrm{D}}^{x,(1-\varepsilon)r}[A_i, B] \leq \widetilde{\mathrm{D}}^{x,r}[A, B] \leq (1+\varepsilon) \liminf_{i\to\infty} \widetilde{\mathrm{D}}^{x,(1+\varepsilon)r}[A_i, B]. \tag{2.13}$$

Proof. Fix $r > 0$. Because $A_i \to A$, we can take $\widetilde{\mathrm{D}}^{0,2r}[A_i, A]$ as small as we like. Thus, we apply the strong quasitriangle inequality to get that for i large enough,

$$\widetilde{\mathrm{D}}^{x,(1-\varepsilon)r}[A_i, B] \leq 2\, \widetilde{\mathrm{D}}^{x,2(1-\varepsilon)r}[A_i, A] + \frac{1}{1-\varepsilon}\, \widetilde{\mathrm{D}}^{x,r}[A, B].$$

Because $\widetilde{\mathrm{D}}^{x,2(1-\varepsilon)r}[A, A_i]$ can be taken to be arbitrarily small, we obtain the first inequality of (2.13). Similarly, we get that for large enough i,

$$\widetilde{\mathrm{D}}^{x,r}[A, B] \leq 2\, \widetilde{\mathrm{D}}^{x,2r}[A, A_i] + (1+\varepsilon)r\, \widetilde{\mathrm{D}}^{x,(1+\varepsilon)r}[A, B].$$

Because $\widetilde{\mathrm{D}}^{x,2r}[A, A_i]$ can be taken to be arbitrarily small, we obtain the second inequality of (2.13). $\square$

2.3 Reifenberg type sets and tangent sets

In this section we develop in high generality methods for studying the local geometry of a set $A \subseteq \mathbb{R}^n$. We begin with a subsection developing local approximability, a framework for comparing A to a nice model class of sets $\mathcal{S}$ in some ball $B(x,r)$. The second subsection covers pseudotangent sets, which capture the infinitesimal behavior of A. In the third subsection, we make precise the connection between the two frameworks.

2.3.1 Local approximability

Definition 2.3.1 (Reifenberg type sets)**.** *Let $A \subseteq \mathbb{R}^n$ and let $x \in A$.*

(i) A local approximation class *$\mathcal{S}$ is a nonempty collection of sets in $\mathfrak{C}(0)$ such that $\mathcal{S}$ is a* cone*; that is, for all $S \in \mathcal{S}$ and $\lambda > 0$, $\lambda S \in \mathcal{S}$.*

(ii) For every $r > 0$, define the approximability *$\Theta_A^{\mathcal{S}}(x,r)$ of A by $\mathcal{S}$ at location x and scale r by*

$$\Theta_A^{\mathcal{S}}(x,r) = \inf_{S \in \mathcal{S}} \widetilde{\mathrm{D}}^{x,r}[A, x+S].$$

(iii) We say A is locally δ-approximable by *$\mathcal{S}$ if for every compact set $K \subseteq A$, there exists a radius r_K such that for all $0 < r \leq r_K$ and $x \in K$, $\Theta_A^{\mathcal{S}}(x,r) \leq \delta$.*

(iv) We say A is locally well approximated by *$\mathcal{S}$ if A is locally δ-approximable by $\mathcal{S}$ for all $\delta > 0$.*

Remark 2.3.2. *Clearly, for all closed sets $A \subseteq \mathbb{R}^n$, locations $x \in A$ and scales $r > 0$, $0 \leq \Theta_A^{\mathcal{S}}(x,r) \leq 1$. Further, we note that in Definition 2.3.1(iii), if we take A to be closed, then we have that A is locally δ-approximable if for any compact set $K \subseteq \mathbb{R}^n$ there exists r_0 such that for all $x \in K \cap A$ and $0 < r \leq r_0$, $\Theta_A^{\mathcal{S}}(x,r) \leq \delta$. Although we typically assume our set A is closed, we will occasionally have reason to discuss local approximability for sets which are not closed (for example, relatively open subsets of closed subsets in Section 4).*

Remark 2.3.3. *Let $A \subseteq \mathbb{R}^n$. We note that A is locally δ-approximable by $\mathcal{S}$ if and only if for each compact subset $K \subseteq A$, there exists an r_K such that $\sup_{x \in K} \sup_{0<r\leq r_K} \Theta_A^{\mathcal{S}}(x,r) \leq \delta$.*

Thus, we get that $\limsup_{r\downarrow 0}\sup_{x\in K}\Theta_A^{\mathcal{S}}(x,r)\le\delta$ *for all compact subsets* K *if and only if* A *is locally* δ'*-approximable by* $\mathcal{S}$ *for all* $\delta'>\delta$.

We now give a lemma about how approximability behaves under limits.

Lemma 2.3.4. *Let* $\mathcal{T}$ *be a local approximation class,* $A, A_i \in \mathfrak{C}(x)$. *If* $A_i \to A$ *in* $\mathfrak{C}(x)$, *then for any* $r>0$ *and* $\varepsilon\in(0,1)$,

$$(1-\varepsilon)\limsup_{i\to\infty}\Theta_{Ai}^{\mathcal{T}}(x,r(1-\varepsilon))\le\Theta_A^{\mathcal{T}}(x,r)\le(1+\varepsilon)\liminf_{i\to\infty}\Theta_{Ai}^{\mathcal{T}}(x,(1+\varepsilon)r)$$

Proof. Follows from Lemma 2.2.6. □

We also give a monotonicity rule for local approximation.

Lemma 2.3.5 (monotonicity)**.** *Suppose* $\mathcal{S}$ *is a local approximation class and* $A\subseteq\mathbb{R}^n$ *is nonempty. If* $B(x,r)\subseteq B(y,s)$ *and* $|x-y|\le ts$, *then*

$$\Theta_A^{\mathcal{S}}(x,r)\le\frac{s}{r}\left(t+(1+t)\Theta_A^{\mathcal{S}}(y,(1+t)s)\right). \tag{2.14}$$

In particular, if $B(x,r)\subseteq B(x,s)$, *then*

$$\Theta_A^{\mathcal{S}}(x,r)\le\frac{s}{r}\Theta_A^{\mathcal{S}}(x,s). \tag{2.15}$$

Proof. Suppose that $B(x,r)\subseteq B(y,s)$ and $|x-y|\le ts$. Let $S\in\mathcal{S}$ be fixed and write $\rho=\widetilde{\mathrm{d}}^{\,y,s}(A,y+S)$. Since $\widetilde{\mathrm{d}}^{\,y,s}(x+S,y+S)\le t$, the strong quasitriangle inequality implies

$$\begin{aligned}\widetilde{\mathrm{D}}^{y,s}[A,x+S]&\le(1+\rho)\,\widetilde{\mathrm{D}}^{y,(1+\rho)s}[x+S,y+S]+(1+t)\,\widetilde{\mathrm{D}}^{y,(1+t)s}[A,y+S]\\&\le t+(1+t)\,\widetilde{\mathrm{D}}^{y,(1+t)s}[A,y+S].\end{aligned}$$

Thus, by monotonicity,

$$\Theta_A^{\mathcal{S}}(x,r)\le\widetilde{\mathrm{D}}^{x,r}[A,x+S]\le\frac{s}{r}\,\widetilde{\mathrm{D}}^{y,s}[A,x+S]\le\frac{s}{r}\Big(t+(1+t)\,\widetilde{\mathrm{D}}^{y,(1+t)s}[A,y+S]\Big).$$

Taking the infimum over $S\in\mathcal{S}$ yields (2.14). □

2.3.2 Tangent sets and pseudotangent sets

Definition 2.3.6 (Tangent Sets and Pseudotangent Sets)**.** *Let $A, T, D \subseteq \mathbb{R}^n$ with A and T closed and let $x \in A$. We say that T is a* pseudotangent set of A at x directed along D *if there exist sequences $x_i \in A$ and $r_i > 0$ such that $x_i \to x$, $r_i^{-1}(x_i - x) \in D$, and $r_i^{-1}(S - x_i) \to T$ in $\mathfrak{C}(0)$. If there is no restriction imposed on D, then we call T a* pseudotangent set *of A at x. If $D = \{0\}$, then we call T a* tangent set of A at x. *We let $\mathrm{Tan}_D(A, x)$, Ψ-$\mathrm{Tan}(A, x)$ and $\mathrm{Tan}(A, x)$ denote the collections of all pseudotangent sets directed along D, all pseudotangent sets, and all tangent sets of A at x, respectively. We say that T is a* tangent set to A at infinity *if there exists a sequence of radii $R_i \to \infty$ such that $A/R_i \to T$. We denote the collection of tangents at infinity by $\mathrm{Tan}(A, \infty)$.*

Since $\mathfrak{C}(0)$ is sequentially compact, $\mathrm{Tan}(A, x) \neq \emptyset$ for all $x \in A$. Moreover, if $(A-x)/r_i \cap D \neq \emptyset$ for some sequence $r_i \downarrow 0$, then $\mathrm{Tan}_D(A, x) \neq \emptyset$. Moreover, if $T \in \Psi$-$\mathrm{Tan}(A, x)$, then $0 \in T$. We now give some useful invariance properties of pseudotangents.

Lemma 2.3.7 (Invariance Properties of Pseudotangents)**.** *Let $A \in \mathfrak{C}(x)$ and $D \subseteq \mathbb{R}^n$.*

(i) *If $B \in \mathrm{Tan}_D(A, x)$ and $\lambda > 0$, then $\lambda B \in \mathrm{Tan}_{\lambda D}(A, x)$.*

(ii) *If $B \in \Psi$-$\mathrm{Tan}(A, x)$, $y \in B$, then $B - y \in \Psi$-$\mathrm{Tan}(A, x)$.*

(iii) *The collection $\mathrm{Tan}_D(A, x)$ is closed.*

(iv) *If $B \in \mathrm{Tan}(A, x)$ and $C \in \mathrm{Tan}(B, 0) \cup \mathrm{Tan}(B, \infty)$, then $C \in \mathrm{Tan}(A, x)$.*

(v) *If $B \in \Psi$-$\mathrm{Tan}(A, x)$ and $C \in \Psi$-$\mathrm{Tan}(B, y)$ for some $y \in B$, then $C \in \Psi$-$\mathrm{Tan}(A, x)$.*

(vi) *If $D \subseteq \mathbb{R}^n$ is compact and $B \in \mathrm{Tan}_D(A, x)$ then there exists $y \in D$ such that $B + y \in \mathrm{Tan}(A, x)$.*

Proof. Fix $A \in \mathfrak{C}(x)$. Without loss of generality, assume that $x = 0$.
(i) Let $B \in \mathrm{Tan}_D(A, 0)$. Then there exist sequences $x_i \in A$ with $x_i \to 0$ and $r_i \downarrow 0$ such that $(A - x_i)/r_i \to B$ and $x_i/r_i \in D$. It is a simple application of the scaling of $\widetilde{\mathrm{D}}^{0,R}$ to check that for any $\lambda > 0$, $(A - x_i)/(r_i\lambda^{-1}) \to \lambda B$. Since $x_i/r_i \in D$, we have that $x_i/(r_i\lambda^{-1}) \in \lambda D$.

(ii) Let $B \in \Psi\text{-Tan}(A, 0)$ an $y \in B$. Then there exist sequences $x_i \in A$ with $x_i \to 0$ and $r_i \downarrow 0$ such that $(A - x_i)/r_i \to B$. Because $y \in B$, there exists a sequence $z_i \in A$ such that $(z_i - x_i)/r_i \to y$. Note that $z_i \to 0$, because $|z_i| \leq r_i|y| + |x_i| \to 0$. We now claim that $(A - z_i)/r_i \to B - y$. Fix $R > 0$. We apply the quasitriangle inequality and monotonicity to get that

$$\begin{aligned}
\widetilde{\mathrm{D}}^{0,\frac{R}{2}}[(A-z_i)/r_i, B-y] &\leq 2\big(\widetilde{\mathrm{D}}^{0,R}[(A-z_i)/r_i, (A-x_i)/r_i-y]+\widetilde{\mathrm{D}}^{0,R}[(A-x_i)/r_i-y, B-y]\big)\\
&\leq 2\left(\frac{1}{R}\big|(z_i - x_i)/r_i - y\big| + \widetilde{\mathrm{D}}^{y,R}[(A-x_i)/r_i, B]\right)\\
&\leq 2\left(\frac{1}{R}\big|(z_i - x_i)/r_i - y\big| + \frac{R+|y|}{R}\,\widetilde{\mathrm{D}}^{0,R+|y|}[(A-x_i)/r_i, B]\right).
\end{aligned}$$

Because $(z - x_i)/r_i \to y$ and $(A - x_i)/r_i \to B$, we have that the above quantity goes to 0. Because R was arbitrary, we have that $(A - z_i)/r_i \to B - y$. Thus $B - y \in \Psi\text{-Tan}(A, 0)$.

(iii) Let $D \subseteq \mathbb{R}^n$. Suppose that $B_i \in \mathrm{Tan}_D(A, x)$ and $B_i \to B$. To prove (iii), we will make a diagonal argument using the increasing radii $R = i$ (where i is the index of our sequence). For each i there exist sequences $x_j^{(i)} \in A$ with $x_j^{(i)} \to 0$ and $r_j^{(i)} \downarrow 0$ such that $(A - x_j^{(i)})/r_j^{(i)} \to B_i$ as $j \to \infty$ and $x_j^{(i)}/r_j^{(i)} \in D$. Thus by choosing j_i large enough, we can ensure that $\widetilde{\mathrm{D}}^{0,i}[(A - x_{ji}^{(i)})/r_{ji}^{(i)}, B_i] < 1/i^2$. Moreover, by possibly increasing j_i, we can ensure that $|x_{ji}^{(i)}| < 1/i$ and $r_{ji}^{(i)} < 1/i$. Let $x_i = x_{ji}^{(i)}$ and $r_i = r_{ji}^{(i)}$.

We now claim that $(A - x_i)/r_i \to B$. Fix $R > 0$. Then for all $i > R$, we apply the quasitriangle inequality and monotonicity to get

$$\begin{aligned}
\widetilde{\mathrm{D}}^{0,R/2}[B, (A - x_i)/r_i] &\leq 2\big(\widetilde{\mathrm{D}}^{0,R}[B, B_i] + \widetilde{\mathrm{D}}^{0,R}[B_i, (A - x_i)/r_i]\big)\\
&\leq 2\big(\widetilde{\mathrm{D}}^{0,R}[B, B_i] + (i/R)\,\widetilde{\mathrm{D}}^{0,i}[B_i, (A - x_i)/r_i]\big)\\
&\leq 2\big(\widetilde{\mathrm{D}}^{0,R}[B, B_i] + 1/(Ri)\big)
\end{aligned}$$

Because $B_i \to B$, we have that the above quantity goes to 0 as $i \to \infty$. Because R was arbitrary, we have that $(A - x_i)/r_i \to B$. By construction, $x_i/r_i \in D$, $x_i \to 0$, and $r_i \downarrow 0$, so $B \in \mathrm{Tan}_D(A, x)$.

(iv) This fact follows by applying (i) and (iii) with $D = \{0\}$.

(v) This fact follows by applying (i), (ii), and (iii) with $D = \mathbb{R}^n$.

(vi) Let $D \subseteq \mathbb{R}^n$ be compact and $B \in \mathrm{Tan}_D(A, x)$. Then there exist $x_i \in A$ with $x_i \to x$ and $r_i \downarrow 0$ such that $(x_i - x)/r_i \in D$. Because D is compact, there is some subsequence (which we do not relabel) such that $(x_i - x)/r_i \to y$ for $y \in D$.

We claim that $(A - x)/r_i \to B + y$. If this holds, then $B + y \in \mathrm{Tan}(A, x)$ whence the statement is proven. Let $R > 0$. For convenience, set $R_y = R + |y|$. By applying monotonicity, translation of $\widetilde{\mathrm{D}}^{z,s}$, and the quasitriangle inequality, we get that

$$\begin{aligned}\widetilde{\mathrm{D}}^{0,R}\left[\frac{A-x}{r_i}, B+y\right] &\leq \frac{R_y}{R}\,\widetilde{\mathrm{D}}^{y,R_y}\left[\frac{A-x}{r_i}, B+y\right] = \frac{R_y}{R}\,\widetilde{\mathrm{D}}^{0,R_y}\left[\frac{A-x}{r_i} - y, B\right] \\ &\leq 2\frac{R_y}{R}\left(\widetilde{\mathrm{D}}^{0,2R_y}\left[\frac{A-x}{r_i} - y, \frac{A-x_i}{r_i}\right] + \widetilde{\mathrm{D}}^{0,2R_y}\left[\frac{A-x_i}{r_i}, B\right]\right) \\ &\leq \frac{1}{R}\left|\frac{x_i - x}{r_i} - y\right| + 2\frac{R_y}{R}\,\widetilde{\mathrm{D}}^{0,2R_y}\left[\frac{A-x_i}{r_i}, B\right].\end{aligned}$$

Because $(x_i - x)/r_i \to y$ and $(A - x_i)/r_i \to B$, we have that the above quantity goes to zero. Because $R > 0$ was arbitrary, the above equation implies that $(A - x)/r_i \to B + y$. □

Remark 2.3.8. *Each statement in Lemma 2.3.7 can be adapted to the setting of tangent and pseudotangent measures, where we replace A with a locally doubling Radon measure μ (see Definitions 4.1.3 and 4.2.1). See [1] Lemma 2.6 for the proof of the analogue to Lemma 2.3.7(iv)).*

Remark 2.3.9. *The collection of* bounded pseudotangents *to A at x is defined to be*

$$\mathrm{b}\Psi\text{-}\mathrm{Tan}(A, x) := \bigcup_{R>0} \mathrm{Tan}_{B(0,R)}(A, x).$$

In this language, Lemma 2.3.7(vi) implies that every bounded pseudotangent is a translate of a tangent set. Conversely, by (ii), every translate of a tangent set is a pseudotangent set. In fact, every translate of a tangent is a bounded pseudotangent set by following carefully the proof of (ii). Thus, the bounded pseudotangent sets are exactly the translations of tangent sets.

We say that an unbounded pseudotangent *of A at x is a set $T \in \Psi\text{-}\mathrm{Tan}(A, x)$ such that $(A - x_i)/r_i \to T$, where $|x - x_i|/r_i \to \infty$. Consider the following example.*

Example. *Let $A = P^{-1}(0)$, where $P : \mathbb{R}^3 \to \mathbb{R}$ is the polynomial*

$$P(x) = x_1^2 + x_2^2 - x_3^2 + x_1^4.$$

Note then that $0 \in A$. *For small* x, x_1^4 *decays much faster than the other terms, and one can show that* $\mathrm{Tan}(A, 0) = \{P_0^{-1}(0)\}$, *where*

$$P_0(x) = x_1^2 + x_2^2 - x_3^2.$$

Consider the pseudotangents of A *at* 0*; suppose that* $(A - x_i)/r_i \to T$. *By potentially passing to a subsequence, it suffices to consider two cases;* x_i/r_i *is bounded, or* $x_i/r_i \to \infty$. *The first case says that* T *is a translate of* $P_0^{-1}(0)$ *by Lemma 2.3.7. In the second case, one can show that* T *is a plane tangent to* $P_0^{-1}(0)$. *Thus, the bounded pseudotangents to* A *at 0 are translates of the cone* $P_0^{-1}(0)$ *and the unbounded pseudotangents to* A *at 0 are planes tangent to* $P_0^{-1}(0)$.

2.3.3 Local approximability vs. pseudotangents

In this section, we give the precise connections between local approximability and pseudotangent sets.

Lemma 2.3.10. *Let* $\mathcal{S}$ *be a local approximation class,* $A \in \mathfrak{C}(x)$. *Then* $\lim_{r \downarrow 0} \Theta_A^{\mathcal{S}}(x, r) = 0$ *if and only if* $\mathrm{Tan}(A, x) \subseteq \overline{\mathcal{S}}$.

Proof. Let $\mathcal{S}$ be a local approximation class and $A \in \mathfrak{C}(x)$. Without loss of generality, assume $x = 0$. Suppose that $\mathrm{Tan}(A, 0) \subseteq \overline{\mathcal{S}}$. Let $r_i \downarrow 0$. By sequential compactness, there exists a subsequence r_{ij} such that $A/r_{ij} \to T$ for some $T \in \mathrm{Tan}(A, 0)$. Thus, $\widetilde{\mathrm{D}}^{0,1}[A/r_{ij}, T] \to 0$. By the scale invariance of $\widetilde{\mathrm{D}}$, we have that $\widetilde{\mathrm{D}}^{0,1}[A/r_{ij}, T] = \widetilde{\mathrm{D}}^{0,r_{ij}}[A, r_{ij}T]$. Because $\mathcal{S}$ is a local approximation class, $r_{ij}T \in \overline{\mathcal{S}}$ and so $\Theta_A^{\mathcal{S}}(0, r_{ij}) \leq \widetilde{\mathrm{D}}^{0,r_{ij}}[A, r_{ij}T] \to 0$. Hence, we conclude that $\lim_{r \downarrow 0} \Theta_A^{\mathcal{S}}(0, r) = 0$.

Coversely, suppose that $\lim_{r \downarrow 0} \Theta_A^{\mathcal{S}}(0, r) = 0$. Suppose that $T \in \mathrm{Tan}(A, 0)$. Then there exists a sequence $r_i \downarrow 0$ such that $A/r_i \to T$. Fix $R > 0$. Because $A/r_i \to T$, we have that $\widetilde{\mathrm{D}}^{0,R}[A/r_i, T] \to 0$. On the other hand, because $\lim_{r \downarrow 0} \Theta_A^{\mathcal{S}}(0, r) = 0$, we have that there exists a sequence $S(i, R) \in \mathcal{S}$ such that $\widetilde{\mathrm{D}}^{0,Rr_i}[A, S(i, R)] \to 0$ as $i \to \infty$. By the scaling of $\widetilde{\mathrm{D}}$, we have that $\widetilde{\mathrm{D}}^{0,R}[A/r_i, S(i, R)/r_i] = \widetilde{\mathrm{D}}^{0,Rr_i}[A, S(i, R)] \to 0$. Let $S'(i, R) = S(i, R)/r_i$. By the quasitriangle inequality, we have that $\widetilde{\mathrm{D}}^{0,R/2}[S'(i, R), T] \leq 2(\widetilde{\mathrm{D}}^{0,R}[S'(i, R), A/r_i] + \widetilde{\mathrm{D}}^{0,R}[A/r_i, T]) \to 0$ for any $R > 0$. Thus we can choose a subsequence $(i_k) \subseteq (i)$ such that

$\widetilde{\mathrm{D}}^{0,k}[S'(i_k,2k),T] \leq 1/k^2$. By monotonicity, for any radius R and $k \geq R$, we have that $\widetilde{\mathrm{D}}^{0,R}[S'(i_k,2k),T] \leq k/R\,\widetilde{\mathrm{D}}^{0,k}[S'(i_k,2k),T] \leq k/R \cdot 1/k^2 = 1/(Rk) \to 0$. This implies that the sequence $S'(i_k,2k) \to T$. Since $S'(i_k,2k) \in \mathcal{S}$, we have that $T \in \overline{\mathcal{S}}$. We conclude that $\mathrm{Tan}(A,0) \subseteq \mathcal{S}$. □

Example. We now give an example to show why Definition 2.3.1(ii) should use the relative Walkup-Wets distance rather than the relative Hausdorff distance. Define

$$S = \{0\} \cup \bigcup_{i\in\mathbb{Z}} \partial B(0,2^i) \subset \mathbb{R}^n$$

and

$$\mathcal{S} = \{\lambda S : \lambda > 0\}.$$

Then $\mathcal{S}$ is a local approximation class, and moreover, $\mathcal{S}$ is closed in $\mathfrak{C}(0)$. Let $r_i = 2^{-i} - 3^{-i}$ for all $i \geq 1$ and let e be a unit vector in $\mathbb{R}^n$. Define $A = S \cup \{r_i e \mid i \geq 1\}$. On one hand, $\Theta^{\mathcal{S}}_A(0,r) \to 0$ as $r \to 0$. Thus, $\mathrm{Tan}(A,0) = \mathcal{S}$ by Lemma 2.3.10. On the other hand, we now claim that $\inf_{S'\in S} \mathrm{D}^{0,r_i}[S',A] \to 1/4$ as $i \to \infty$. To see this we consider an element $S' = \lambda S$. Because $2S = S$, we may assume that $\lambda \in (r_i/2, r_i]$. We now note that $d(r_i e, \lambda S) = r_i - \lambda$. We note also that $\lambda\partial B(0,1) = \partial B(0,\lambda) \subseteq B(0,r_i)$. We compute that $d(\partial B(0,\lambda), A) = d(\partial B(0,\lambda), \partial B(0,2^{i+1})) = \lambda - 2^{i+1}$. Thus, we have a lower bound for $\mathrm{D}^{0,r_i}[S',A]$ of $\max(r_i - \lambda, \lambda - 2^{i+1})/r_i$. Because $\lambda \in (r_i/2, r_i]$, we may substitute $\gamma = \lambda/r_i \in (1/2,1]$ to get that $\mathrm{D}^{0,r_i}[S',A] \geq \max(1-\gamma, \gamma - 2^{i+1}/r_i)$. As $i \to \infty$, this bound goes to $\max(1-\gamma, \gamma - 1/2)$ for $\gamma \in [1/2,1]$. This is infimized at $1/4$ with $\gamma = 1/4$. Thus, we get a lower bound for the limit of $\inf_{S'\in S} \mathrm{D}^{0,r_i}[S',A]$ of $1/4$. It can be shown moreover that this is optimal.

Lemma 2.3.11. *Let $\mathcal{S}$ be a local approximation class, $A \in \mathfrak{C}(x)$. Then $\liminf_{r\downarrow 0} \Theta^{\mathcal{S}}_A(x,r) = 0$ if and only if $\mathrm{Tan}(A,x) \cap \overline{\mathcal{S}} \neq \emptyset$.*

Proof. The proof of Lemma 2.3.11 is similar to the proof of Lemma 2.3.10 using infimizing sequences. □

Pseudotangents may be used to show that a set $A \subseteq \mathbb{R}^n$ is locally well approximated by $\mathcal{S}$. We here make precise this connection similarly to Lemma 2.3.10.

Lemma 2.3.12. *Let $\mathcal{S}$ be a local approximation class, let $A \subseteq \mathbb{R}^n$ be a nonempty set. Then A is locally well approximated by $\mathcal{S}$ if and only if $\Psi\text{-Tan}(A,x) \subseteq \overline{\mathcal{S}}$ for all $x \in A$.*

Proof. Suppose that $\Psi\text{-Tan}(A,x) \subseteq \overline{\mathcal{S}}$ for all $x \in A$. Let $K \subseteq A$ be a compact set. Set $\ell = \limsup_{r\downarrow 0}\sup_{x\in K}\Theta^{\mathcal{S}}_A(x,r)$. Then there exist sequences $x_i \in K$, $r_i \downarrow 0$ such that $\Theta(x_i,r_i) \to \ell$. Because K is compact, there is a subsequence $(i_j) \subseteq (i)$ such that $x_{ij} \to x$ for some $x \in K$. By sequential compactness of $\mathfrak{C}(0)$, there is a subsequence $(i_k) \subseteq (i_j)$ such that $(A - x_{ik})/r_{ik} \to T$ for some $T \in \Psi\text{-Tan}(A,x)$. By assumption, $T \in \overline{\mathcal{S}}$. Thus, $\Theta^S_A(x_i,r_i) \leq \widetilde{\mathrm{D}}^{x_i,r_i}[A, r_iS + x_i] = \widetilde{\mathrm{D}}^{0,1}[(A-x_i)/r_i, S] \to 0$. Hence, $\ell = 0$. Because $K \subseteq A$ was an arbitrary compact set, we establish that $\Theta^{\mathcal{S}}_A(x,r) \to 0$ uniformly on compact subsets of A. So by definition, A is locally well approximated by $\mathcal{S}$.

Suppose that A is locally well approximated by $\mathcal{S}$. Let $x \in A$ and $T \in \Psi\text{-Tan}(A,x)$. Then there exist sequences $x_i \in A$ with $x_i \to x$ and $r_i \downarrow 0$ such that $(A-x_i)/r_i \to T$. Fix $R > 0$. Because $(A-x_i)/r_i \to T$, we have that $\widetilde{\mathrm{D}}^{0,R}[(A-x_i)/r_i, T] \to 0$. Let $X := \{x_i\}_{i=1}^\infty \cup \{x\}$. Then X is compact, so by assumption $\lim_{r\downarrow 0}\sup_{y\in X}\Theta^{\mathcal{S}}_A(y,r) = 0$. Thus, we have that there exists a sequence $S(i,R) \in \mathcal{S}$ such that $\widetilde{\mathrm{D}}^{x_i,Rr_i}[A, S(i,R)+x_i] \to 0$ as $i \to \infty$. By the scaling of $\widetilde{\mathrm{D}}$, we have that $\widetilde{\mathrm{D}}^{0,R}[(A-x_i)/r_i, S(i,R)/r_i] = \widetilde{\mathrm{D}}^{0,Rr_i}[A, S(i,R)] \to 0$. Let $S'(i,R) = S(i,R)/r_i$. By the quasitriangle inequality, we have that $\widetilde{\mathrm{D}}^{0,R/2}[S'(i,R),T] \leq 2(\widetilde{\mathrm{D}}^{0,R}[S'(i,R),(A-x_i)/r_i] + \widetilde{\mathrm{D}}^{0,R}[A/r_i,T]) \to 0$ for any $R > 0$. Thus we can choose a subsequence $(i_k) \subseteq (i)$ such that $\widetilde{\mathrm{D}}^{0,k}[S'(i_k,2k),T] \leq 1/k^2$. By monotonicity, for any radius R and $k \geq R$, we have that $\widetilde{\mathrm{D}}^{0,R}[S'(i_k,2k),T] \leq k/R\,\widetilde{\mathrm{D}}^{0,k}[S'(i_k,2k),T] \leq k/R \cdot 1/k^2 = 1/(Rk) \to 0$. This implies that the sequence $S'(i_k,2k) \to T$. Because $S'(i_k,2k) \in \mathcal{S}$, we have that $T \in \overline{\mathcal{S}}$. We conclude that $\Psi\text{-Tan}(A,x) \subseteq \overline{\mathcal{S}}$. □

We now give a perturbation of Lemmas 2.3.10 and 2.3.12. For $\varepsilon \geq 0$, define the local approximation class

$$(\mathcal{S};\varepsilon)_{0,\infty} = \{\hat{S} : 0 \in \hat{S} \text{ and } \Theta^{\mathcal{S}}_{\hat{S}}(0,r) \leq \varepsilon \text{ for all } r > 0\}.$$

We remark that $(\mathcal{S};\varepsilon)_{0,\infty}$ is closed for all $\varepsilon \geq 0$ and $(\mathcal{S};0)_{0,\infty} = \overline{\mathcal{S}}$.

Similarly, for $\varepsilon \geq 0$, define the local approximation class

$$(\mathcal{S};\varepsilon)_{\mathbb{R}^n,\infty} = \{\hat{S} : 0 \in \hat{S} \text{ and } \Theta^{\mathcal{S}}_{\hat{S}}(x,r) \leq \varepsilon \text{ for all } x \in \hat{S} \text{ and all } r > 0\}.$$

We remark that $(\mathcal{S};\varepsilon)_{\mathbb{R}^n,\infty}$ is closed for all $\varepsilon \geq 0$. Moreover, if $\mathcal{S}$ is translation invariant (that is, $S \in \mathcal{S}$ and $x \in S$ implies $S - x \in \mathcal{S}$), then we also have that $(\mathcal{S};0)_{\mathbb{R}^n,\infty} = \overline{\mathcal{S}}$. If $\mathcal{S}$ is not translation invariant, then $(\mathcal{S};0)_{\mathbb{R}^n,\infty} \subset \overline{\mathcal{S}}$ (see Lemma 2.3.7(ii) for comparison).

Example. Consider $\mathcal{G} = \mathcal{G}(n,m)$, the Grassmanian of m-planes in $\mathbb{R}^n$. Because $\mathcal{G}$ is closed, $(\mathcal{G};0)_{0,\infty} = \mathcal{G}$. For $\varepsilon > 0$, $(\mathcal{G};\varepsilon)_{\mathbb{R}^n,\infty}$ is the collection of globally ε-Reifenberg flat sets, as defined by Kenig and Toro (see for example [18]). These sets have particularly nice properties, such as being tame bi-Hölder images of $\mathbb{R}^m$ for ε small enough (see David and Toro's work on Reifenberg flat metric spaces [12]), and having Minkowski dimension bounded by $m + C_{n,m}\varepsilon^2$ by the work of Mattila and Vuorinen [26].

Consider the cone $C = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = x_3^2\}$, and let $\mathcal{C}$ be the local approximation class containing all rotations of C followed by translations $C - y$ such that $y \in C$. It follows that $\mathcal{C}$ is a local approximation class. Because C is smooth away from the origin, it follows that ε Reifenberg flat sets are in $(\mathcal{C};\varepsilon)$. Moreover, using the technology of detectability which we develop in the coming sections, one can show that $(\mathcal{C};\varepsilon) = \widetilde{\mathcal{C}} \cup (\mathcal{G};\varepsilon)$, where $(\mathcal{G};\varepsilon)$ is the collection of ε-Reifenberg flat sets and each $\widetilde{C} \in \widetilde{\mathcal{C}}$ is a global deformation of C.

Lemma 2.3.13. *Let $\mathcal{S}$ be a local approximation class, let $A \subset \mathbb{R}^n$, let $x \in A$, and let $\varepsilon > 0$. Then $\limsup_{r\downarrow 0} \Theta_A^{\mathcal{S}}(x,r) \leq \varepsilon$ if and only if $\operatorname{Tan}(A,x) \subset (\mathcal{S};\varepsilon)_{0,\infty}$.*

Proof. Suppose that $\limsup_{r\downarrow 0} \Theta_A^{\mathcal{S}}(x,r) \leq \varepsilon$. Let $T \in \operatorname{Tan}(A,x)$. Then there exists a sequence $r_i \downarrow 0$ such that $(A-x)/r_i \to T$. We show that for any $r > 0$, $\Theta_T^{\mathcal{S}}(0,r) \leq \varepsilon$, and thus $T \in (\mathcal{S};\varepsilon)_{0,\infty}$. Fix $r > 0$ and $\delta > 0$. By assumption, there exists I large enough such that for all $i \geq I$, $\widetilde{\mathrm{D}}^{0,2r}[(A-x)/r_i, T] < \delta$. By monotonicity, we also have that $\widetilde{\mathrm{D}}^{0,r}[(A-x)/r_i, T] < 2\delta$. By supposition, there exists I large enough such that for $i \geq I$, there exists $S_i \in \mathcal{S}$ such that $\widetilde{\mathrm{D}}^{0,(1+2\delta)r}[(A-x)/r_i, S_i] < \varepsilon + \delta$. By monotonicity, we also have that $\widetilde{\mathrm{D}}^{0,r}[(A-x)/r_i, S_i] < (1+2\delta)(\varepsilon+\delta)$. We apply the strong quasitriangle inequality to get that

$$\begin{aligned}\widetilde{\mathrm{D}}^{0,r}[S_i, T] &\leq (1+2\delta)\,\widetilde{\mathrm{D}}^{0,(1+2\delta)r}[S_i, (A-x)/r_i] + 2\,\widetilde{\mathrm{D}}^{0,2r}[(A-x)/r_i, T] \\ &\leq (1+2\delta)(\varepsilon+\delta) + 2\delta.\end{aligned}$$

Because this holds for all $\delta > 0$, we get that $\Theta_T^{\mathcal{S}}(0,r) \leq \varepsilon$.

Conversely, suppose that $\operatorname{Tan}(A,x) \subseteq (S;\varepsilon)_{0,\infty}$. Let $r_i \downarrow 0$ be a sequence such that $\Theta_A^{\mathcal{S}}(x,r_i) \to \limsup_{r\downarrow 0} \Theta_A^{\mathcal{S}}(x,r)$. By sequential compactness there exists a subsequence (which we do not relabel) such that $(A-x)/r_i \to T$ for some $T \in \operatorname{Tan}(A,x)$. Fix $\delta > 0$. There exists I large enough so that for all $i \geq I$, $\widetilde{\mathrm{D}}^{0,2}[(A-x)/r_i, T] < \delta$. By supposition, $T \in (S;\varepsilon)_{0,\infty}$, so there exists $S \in \mathcal{S}$ such that $\widetilde{\mathrm{D}}^{0,1+2\delta}[T,S] \leq \varepsilon + \delta$. Similarly to before, we apply monotonicity and the strong quasitriangle inequality to get that

$$\begin{aligned}\widetilde{\mathrm{D}}^{0,1}[(A-x)/r_i, S] &\leq 2\,\widetilde{\mathrm{D}}^{0,2r}[(A-x)/r_i, T] + (1+2\delta)\,\widetilde{\mathrm{D}}^{0,1+2\delta}[T,S] \\ &\leq (1+2\delta)(\varepsilon+\delta) + 2\delta.\end{aligned}$$

Applying scale invariance, we get that for i large enough,

$$\widetilde{\mathrm{D}}^{x,r_i}[A, r_iS + x] = \widetilde{\mathrm{D}}^{0,1}[(A-x)/r_i, S] \leq (1+2\delta)(\varepsilon+\delta) + 2\delta.$$

Because $\delta > 0$ was arbitrary, it follows that $\limsup_{r\downarrow 0} \Theta_A^{\mathcal{S}}(x,r) \leq \varepsilon$. □

Lemma 2.3.14. *Let $\mathcal{S}$ be a local approximation class, let $A \subseteq \mathbb{R}^n$ be a nonempty set and let $\varepsilon > 0$. Then A is locally ε'-approximable by $\mathcal{S}$ for all $\varepsilon' > \varepsilon$ if and only if* Ψ-$\operatorname{Tan}(A,x) \subseteq (\mathcal{S};\varepsilon)_{\mathbb{R}^n,\infty}$ *for all $x \in A$.*

Proof. Fix $\varepsilon > 0$. Suppose that A is locally ε'-approximable by $\mathcal{S}$ for all $\varepsilon' > \varepsilon$. Then for every compact subset $K \subseteq A$, $\limsup_{r\downarrow 0} \sup_{x\in K} \Theta_A^{\mathcal{S}}(x,r) \leq \varepsilon$ (see Remark 2.3.3). Let $T \in \Psi$-$\operatorname{Tan}(A,x)$. Then there exist sequences $x_i \in A$ with $x_i \to x$ and $r_i \downarrow 0$ such that $(A-x_i)/r_i \to T$. We begin by showing that for any $r > 0$, $\Theta_T^{\mathcal{S}}(0,r) \leq \varepsilon$. Fix $r > 0$ and $\delta > 0$. By assumption, there exists I large enough such that for all $i \geq I$, $\widetilde{\mathrm{D}}^{0,2r}[(A-x_i)/r_i, T] < \delta$. By monotonicity, we also have that $\widetilde{\mathrm{D}}^{0,r}[(A-x_i)/r_i, T] < 2\delta$. By supposition, there exists I large enough such that for $i \geq I$, there exists $S_i \in \mathcal{S}$ such that $\widetilde{\mathrm{D}}^{0,(1+2\delta)r}[(A-x_i)/r_i, S_i] < \varepsilon + \delta$. By monotonicity, we also have that $\widetilde{\mathrm{D}}^{0,r}[(A-x_i)/r_i, S_i] < (1+2\delta)(\varepsilon+\delta)$. We apply the strong quasitriangle inequality to get that

$$\begin{aligned}\widetilde{\mathrm{D}}^{0,r}[S_i, T] &\leq (1+2\delta)\,\widetilde{\mathrm{D}}^{0,(1+2\delta)r}[S_i, (A-x_i)/r_i] + 2\,\widetilde{\mathrm{D}}^{0,2r}[(A-x_i)/r_i, T] \\ &\leq (1+2\delta)(\varepsilon+\delta) + 2\delta.\end{aligned}$$

Because this holds for all $\delta > 0$, we get that for all pseudotangents T and all $r > 0$, $\Theta_T^{\mathcal{S}}(0,r) \leq \varepsilon$. Next, we apply Lemma 2.3.7(ii) to get that for all $y \in T$, $T - y$ is a pseudotangent as well. Thus, $\Theta^{\mathcal{S}}_{T-y}(0,r) \le \varepsilon$ for all $y \in T$ and $r > 0$. By translating, we get that $\Theta^{\mathcal{S}}_{T}(y,r) \le \varepsilon$ for all $y \in T$ and $r > 0$. Thus $T \in (S;\varepsilon)_{\mathbb{R}^n,\infty}$.

Conversely, suppose that $\Psi\text{-Tan}(A,x) \subseteq (S;\varepsilon)_{0,\infty}$ for all $x \in A$. Fix a compact set $K \subseteq A$. Let $x_i \in K$, $r_i \downarrow 0$ be sequences such that $\Theta^{\mathcal{S}}_{A}(x_i,r_i) \to \limsup_{r\downarrow 0}\sup_{x\in K}\Theta^{\mathcal{S}}_{A}(x,r)$. By sequential compactness there exist subsequences (which we do not relabel) such that $x_i \to x$ for some $x \in K$ and $(A-x_i)/r_i \to T$ for some $T \in \Psi\text{-Tan}(A,x)$. Fix $\delta > 0$. There exists I large enough so that for all $i \ge I$, $\widetilde{\mathrm{D}}^{0,2}[(A-x_i)/r_i, T] < \delta$. By supposition, $T \in (S;\varepsilon)_{\mathbb{R}^n,\infty}$, so there exists $S \in \mathcal{S}$ such that $\widetilde{\mathrm{D}}^{0,1+2\delta}[T,S] \le \varepsilon + \delta$. Similarly to before, we apply monotonicity and the strong quasitriangle inequality to get that

$$\begin{aligned}\widetilde{\mathrm{D}}^{0,1}[(A-x_i)/r_i, S] &\le 2\,\widetilde{\mathrm{D}}^{0,2r}[(A-x)/r_i, T] + (1+2\delta)\,\widetilde{\mathrm{D}}^{0,1+2\delta}[T,S] \\ &\le (1+2\delta)(\varepsilon+\delta) + 2\delta.\end{aligned}$$

Applying scale invariance, we get that for i large enough,

$$\widetilde{\mathrm{D}}^{x_i,r_i}[A, r_iS + x] = \widetilde{\mathrm{D}}^{0,1}[(A-x_i)/r_i, S] \le (1+2\delta)(\varepsilon+\delta) + 2\delta.$$

Because $\delta > 0$ was arbitrary, it follows that $\limsup_{r\downarrow 0}\Theta^{\mathcal{S}}_{A}(x,r) \le \varepsilon$. □

2.4 Connectedness of the cone of tangent sets at a point

In this section, we prove that the cone of tangent sets at point is connected at infinity. Roughly speaking, this means that if $\mathrm{Tan}(A,x)$ is contained in a local approximation class $\mathcal{S}$ which has two pieces separated at large scales, then $\mathrm{Tan}(A,x)$ only intersects one of those pieces. This result is motivated by an analogous statement for tangent measures established by Preiss [28] Theorem 2.6. Also see Kenig, Preiss and Toro [17] Theorem 2.1.

Definition 2.4.1 (Separation at infinity)**.** *Let $\mathcal{S}$ and $\mathcal{T}$ be local approximation classes such that $\mathcal{T} \subset \mathcal{S}$. We say that $\mathcal{T}$* is separated at infinity in $\mathcal{S}$ *if*

$$\inf_{S\in\mathcal{S}\setminus\mathcal{T}} \limsup_{r\uparrow\infty} \Theta^{\mathcal{T}}_{S}(0,r) \ge \phi \quad \textit{for some } \phi > 0.$$

To emphasize a choice of some $\phi > 0$, we may say that $\mathcal{T}$ is ϕ-separated at infinity in $\mathcal{S}$.

Theorem 2.4.2 (Connectedness of the cone of tangent sets at a point)**.** *Let $\mathcal{S}$ and $\mathcal{T}$ be local approximation classes such that $\mathcal{T} \subset \mathcal{S}$. If $\mathcal{T}$ is separated at infinity in $\mathcal{S}$, then for all closed sets $A \subseteq \mathbb{R}^n$ and $x \in A$,*

$$\operatorname{Tan}(A,x) \subseteq \mathcal{S} \text{ and } \operatorname{Tan}(A,x) \cap \mathcal{T} \neq \emptyset \Longrightarrow \operatorname{Tan}(A,x) \subseteq \overline{\mathcal{T}}.$$

Proof. Let $\mathcal{T} \subseteq \mathcal{S}$ be local approximation classes. Let $A \subseteq \mathbb{R}^n$ and $x \in A$. Without loss of generality, assume that $x = 0$. Assume that $\mathcal{T}$ is ϕ-separated at infinity in $\mathcal{S}$; that is,

$$\inf_{S \in \mathcal{S} \setminus \overline{\mathcal{T}}} \limsup_{r \to \infty} \Theta_S^{\mathcal{T}}(0,r) > \phi. \tag{2.16}$$

Suppose for contradiction that $\operatorname{Tan}(A,0) \subseteq \mathcal{S}$, $\operatorname{Tan}(A,0) \cap \mathcal{T} \neq \emptyset$ and $\operatorname{Tan}(A,0) \cap \mathcal{S} \setminus \overline{\mathcal{T}} \neq \emptyset$. Let $T \in \operatorname{Tan}(A,0) \cap \mathcal{T}$ and $R \in \operatorname{Tan}(A,0) \cap R \in \mathcal{S} \setminus \overline{\mathcal{T}}$. Then there exist sequences $t_i \downarrow 0$ and $r_i \downarrow 0$ such that $A/t_i \to T$ and $A/r_i \to R$. Without loss of generality, we assume that $t_{i+1} < 2r_i < t_i$ and $t_1 < 1$.

By (2.16) and because $R \notin \overline{\mathcal{T}}$, there exists $c > 0$ such that $\Theta_R^{\mathcal{T}}(0,c) > \phi$. For all $t > 0$, let us abbreviate

$$\Theta_{t^{-1}A}^{\mathcal{T}}(0,c) = \Theta_A^{\mathcal{T}}(0,tc) =: \Theta(t). \tag{2.17}$$

Note that $\Theta(t_i) \leq \widetilde{\mathrm{D}}^{0,c}[A/t_i, T] \to 0$. Next, we argue that for i large enough, $\Theta(2r_i) > \phi/2$. We see this as follows. Let $T' \in \mathcal{T}$. By the quasitriangle inequality, we have that $\phi < \widetilde{\mathrm{D}}^{0,c}[R,T'] \leq 2(\widetilde{\mathrm{D}}^{0,2c}[R, A/r_i] + \widetilde{\mathrm{D}}^{0,2c}[A/r_i, T'])$. Because $A/r_i \to R$ we have that $\widetilde{\mathrm{D}}^{0,2c}[R, A/r_i] \to 0$, so for i large enough, $\phi < 2\,\widetilde{\mathrm{D}}^{0,2c}[A/r_i, T']$. We rearrange and apply the scaling of $\widetilde{\mathrm{D}}$ to get that $\phi/2 < \widetilde{\mathrm{D}}^{0,c}[A/(2r_i), T'/2]$. Because $\mathcal{T}$ is a cone and $T' \in \mathcal{T}$ was arbitrary, we get that $\phi/2 < \Theta(2r_i)$.

Note that $\Theta(t) \in [0,1]$. Further, we now argue that $\Theta(t) \neq 0$ for any $t \in (0,1]$. Suppose that there were such a $t \in (0,1]$. Then by monotonicity, for any $s \in (0,1]$, we see that $\Theta_A^{\mathcal{T}}(0,stc) \leq s^{-1}\Theta_A^{\mathcal{T}}(0,tc) = 0$. Hence $\operatorname{Tan}(A,0) \subseteq \overline{\mathcal{T}}$ by Lemma 2.3.10, contradicting our supposition. Thus, $\Theta(t) \in (0,1]$ for all $t \in (0,1]$. Set $I_j = (2^{-j}, 2^{-j+1}]$ for all $j \geq 1$ and set $I_j^+ = \cup_{k \geq j} I_j = (0, 2^{-j+1}]$. For all $t \in (0,1]$, there exists a unique j with $\Theta(t) \in I_j$. Let p be the unique integer such that $\phi \in I_p$.

We now aim to construct a sequence $s_i \to 0$ such that A/s_i converges to some tangent set of A which cannot possibly lie in either $\overline{\mathcal{T}}$ or $\mathcal{S} \setminus \overline{\mathcal{T}}$. First, we note an easy consequence of monotonicity, the *weak jump bound*:

$$\Theta(t) \in I_k \Rightarrow \Theta(st) \in I^+_{k-1} \text{ for all } s \in [1/2, 1] \text{ and } t \in (0, 1]. \tag{2.18}$$

Claim. *For all i sufficiently large, there exists $s_i \in (2r_i, t_i)$ such that $\Theta(s_i) \in I_{p+3}$, $\Theta(t) \in I^+_{p+3}$ for all $t \in [s_i, t_i]$, and $t_i/s_i \to \infty$.*

Proof of claim. Choose i_0 large enough so that for $i \geq i_0$, $\Theta(2r_i) > \phi/2$ and $\Theta(t_i) < \phi/16$. Let k_i be the unique integer such that $\Theta(t_i) \in I_{k_i}$. In particular, we know that because $\phi \in I_p$, $\phi/2 \in I_{p+1}$ and $\Theta(t_i) \in I^+_{p+4}$. Fix $i \geq i_0$, and consider the sequence $2^{-m}t_i$ for $m \geq 0$. Let M be the largest integer such that $2r_i \leq 2^{-M}t_i$. In particular, the weak jump bound (2.18) implies that $\phi/2 < \Theta(2r_i) \leq 2\Theta(2^{-M}t_i)$, and so $\phi/4 < \Theta(2^{-M}t_i)$. Thus, by the weak jump bound (2.18), we have that there exists $m \in \{0, 1, \ldots, M\}$ such that $\Theta(2^{-m}t_i) \in I_{p+3}$. Let m_i be the least such integer with this property, and let $s_i = 2^{-m_i}t_i$. By construction, $\Theta(s_i) \in I_{p+3}$. Let $t \in (s_i, t_i]$. There exists an integer m with $0 \leq m < m_i$ with $t \in (2^{-m-1}t_i, 2^{-m}t_i]$. By the minimality of m_i, $\Theta(2^{-m}t_i) \in I^+_{p+4}$, and so by the weak jump bound (2.18), $\Theta(t) \in I^+_{p+3}$. Finally, we note that $t_i/s_i = 2^{m_i}$. By the weak jump bound (2.18), $m_i \geq k_i - (p+2)$. Because $\Theta(t_i) \to 0$, we have that $k_i \to \infty$, and so $t_i/s_i \to \infty$. This completes the proof of the claim. □

We now complete the proof of the theorem. Passing to a subsequence, we may assume that there exists a set $S \in \mathcal{S}$ such that $A/s_i \to S$ in $\mathfrak{C}(0)$. On one hand, the claim tells us that $\Theta(s_i) = \Theta_{A/s_i}(0, c) \in (2^{-(p+3)}, 2^{-(p+2)}]$. Thus, Lemma 2.3.4 tell us that

$$\Theta^{\mathcal{T}}_S(0, c) \in (2^{-(p+4)}, 2^{-(p+1)}]. \tag{2.19}$$

Hence $S \in \mathcal{S} \setminus \overline{\mathcal{T}}$. On the other hand, suppose that $r > c$. By the claim and Lemma 2.3.4,

$$\Theta^{\mathcal{T}}_S(0, r) \leq 2 \limsup_{i\to\infty} \Theta(2(r/c)s_i) \leq 2^{-(p+1)} < \phi \tag{2.20}$$

since $\Theta(t) \in I^+_{p+3}$ for all $t \in [s_i, t_i]$ and $\liminf_{i\to\infty} t_i/s_i > 2r/c$. Because the choice of $r > c$ was arbitrary, $\limsup_{r\to\infty} \Theta^{\mathcal{T}}_S(0, r) < \phi$, a contradiction to ϕ-separation at infinity. Thus, we conclude that if $\operatorname{Tan}(A, 0) \subseteq \mathcal{S}$ and $\operatorname{Tan}(A, 0) \cap \mathcal{T} \neq \emptyset$, then $\operatorname{Tan}(A, 0) \subseteq \overline{\mathcal{T}}$. □

We end the section with a few criteria for checking separation at infinity.

Lemma 2.4.3. *Let $\mathcal{T} \subseteq \mathcal{S}$ be local approximation classes and let $\phi > 0$. If for all $S \in \mathcal{S}$ there is a function $\Phi_S : (0,1) \to (0,\infty)$ with $\lim_{s\to 0+} \Phi_S(s) = 0$ such that*

$$\Theta_S^{\mathcal{T}}(0,r) < \phi \Longrightarrow \Theta_S^{\mathcal{T}}(0,sr) < \Phi_S(s) \text{ for all } s \in (0,1),$$

then $\inf_{S\in\mathcal{S}\setminus\overline{\mathcal{T}}} \liminf_{r\uparrow\infty} \Theta_S^{\mathcal{T}}(0,r) \geq \phi$, and thus, $\mathcal{T}$ is ϕ separated at infinity in $\mathcal{S}$.

Proof. Suppose for contradiction that there exists $S \in \mathcal{S} \setminus \overline{\mathcal{T}}$ and a sequence $r_i \to \infty$ such that $\Theta_S^{\mathcal{T}}(0,r_i) < \phi$ for all $i \geq 1$. Because $S \notin \overline{\mathcal{T}}$, there exists $j \geq 1$ such that $\delta = \Theta_S^{\mathcal{T}}(0,r_j) > 0$. Since $\lim_{s\to 0+} \Phi_S(s) = 0$, we can find $k \geq j$ such that $\Phi(r_j/r_k) \leq \delta$. Hence $\Theta_S^{\mathcal{T}}(0,r_j) = \Theta_S^{\mathcal{T}}(0,(r_j/r_k)r_k) < \Phi_S(r_j/r_k) \leq \Theta_S^{\mathcal{T}}(0,r_j)$, which is absurd. □

The following property, which we call detectability, is a uniform version of the criterion in Lemma 2.4.3 for separation at infinity. In the next section, we shall see that separation at infinity is effective at giving pointwise information about the tangents of sets, whereas detectability is effective at giving locally uniform information about a set.

Definition 2.4.4 ($\mathcal{T}$ point detection property). *Let $\mathcal{T} \subset \mathcal{S}$ be local approximation classes. We say that $\mathcal{T}$* points are detectable in *$\mathcal{S}$ if there exist a constant $\phi > 0$ and a function $\Phi : (0,1) \to (0,\infty)$ with $\lim_{s\to 0+} \Phi(s) = 0$ such that if $S \in \mathcal{S}$ and $\Theta_S^{\mathcal{T}}(0,r) < \phi$, then $\Theta_S^{\mathcal{T}}(0,sr) < \Phi(s)$ for all $s \in (0,1)$. To emphasize a choice of ϕ and Φ, we may say that $\mathcal{T}$* points are (ϕ,Φ) detectable in *$\mathcal{S}$.*

We now show that if a set is well approximated by $\mathcal{S}$ and $\mathcal{T}$ is detectable in $\mathcal{S}$, then the $\mathcal{T}$ points satisfy a local version of $\mathcal{T}$ detectability. The first version of this Lemma is due to Badger and appeared in [2] in the context of harmonic analysis. Badger considered the local approximation class $\mathcal{H}_d$ whose elements are the zero sets of harmonic polynomials with degree d or less. He showed that $\mathcal{G} = \mathcal{G}(n,n-1)$ is $(\phi,Cs)-$detectable in $\mathcal{H}_d$, and proved Lemma 2.4.5 for $\Phi(s) = Cs$.

Lemma 2.4.5. *Suppose that $\mathcal{S},\mathcal{T}$ are local approximation classes, $\mathcal{T} \subset \mathcal{S}$, and $\mathcal{T}$ is (ϕ,Φ) detectable in $\mathcal{S}$. For all $\delta > 0$ there exist $\varepsilon = \varepsilon(\phi,\Phi,\delta) > 0$ and $\eta = \eta(\phi,\Phi,\delta) > 0$ with the following property. Assume that $A \subseteq \mathbb{R}^n$ is closed, $x \in A$, $r > 0$, and*

$$\sup_{0<r'\leq r} \Theta_A^{\mathcal{S}}(x,r') < \varepsilon. \tag{2.21}$$

If $\Theta^{\mathcal{T}}_A(x,r) < \eta$, *then* $\sup_{0<r'\le r} \Theta^{\mathcal{T}}_A(x,r') < \delta$.

Proof. Assume that $\mathcal{T}$ is (ϕ,Φ)-detectable in $\mathcal{S}$. Let $\delta > 0$ be given and fix parameters $\varepsilon > 0$, $\sigma > 0$, and $\tau > 0$ to be chosen later. Let $A \in \mathfrak{C}(x)$ for some x and assume that (2.21) holds for some $r > 0$. Without loss of generality, we assume that $x = 0$.

Suppose that $\Theta^{\mathcal{T}}_A(0,r) < \tau$. By definition, there exists a set $T \in \mathcal{T}$ such that

$$\widetilde{\mathrm{D}}^{0,r}[A,T] < \tau.$$

On the other hand, since $\Theta^{\mathcal{S}}_A(0,r) < \varepsilon$, there exists a set $S \in \mathcal{S}$ such that

$$\widetilde{\mathrm{D}}^{0,r}[A,S] < \varepsilon. \tag{2.22}$$

Thus, by the quasitriangle inequality,

$$\widetilde{\mathrm{D}}^{0,r/2}[T,S] < 2(\tau+\varepsilon). \tag{2.23}$$

Choose $0 < s < 1/4$ small enough such that $\Phi(4s) < \sigma$. Assume that $2(\tau+\varepsilon) < \phi$. Since $\mathcal{T}$ is (ϕ,Φ) detectable in $\mathcal{S}$, by (2.23) we have $\Theta^{\mathcal{T}}_S(0,2sr) = \Theta^{\mathcal{T}}_S(0,4s(r/2)) < \Phi_{\mathcal{S}}(4s) < \sigma$. Hence, there exists $U \in \mathcal{T}$ such that

$$\widetilde{\mathrm{D}}^{0,2sr}[S,U] < \sigma. \tag{2.24}$$

Because $2s < 1$, it follows from (2.22) and monotonicity that

$$\widetilde{\mathrm{D}}^{0,2sr}[A,S] < \varepsilon/2s. \tag{2.25}$$

By the quasitriangle inequality, (2.24) and (2.25) imply that

$$\widetilde{\mathrm{D}}^{0,sr}[A,U] < 2(\sigma+\varepsilon/2s) = 2\sigma+\varepsilon/s. \tag{2.26}$$

Hence we have proved that

$$\Theta^{\mathcal{T}}_A(0,sr) < 2\sigma+\varepsilon/s \;\text{ provided } 2(\tau+\varepsilon) < \phi_{\mathcal{S}}, s \in (0,1/4), \text{ and } \Phi(4s) < \sigma. \tag{2.27}$$

We are ready to choose parameters. Set $\tau = \min(\delta,\phi)/3$, $\sigma = \tau/4$, s small enough so that $\Phi_{\mathcal{S}}(4s) < \sigma$, and $\varepsilon = s\tau/2$. First, we note that $2(\tau+\varepsilon) < 2(\tau+\tau/4) < 5/2 \cdot \phi_{\mathcal{S}}/3 < \phi_{\mathcal{S}}$. Thus, the hypotheses of (2.27) are met, and it follows that

$$\Theta^{\mathcal{T}}_A(0,sr) < 2\sigma+\varepsilon/s = \tau/2+\tau/2 = \tau. \tag{2.28}$$

To summarize thus far, we have proven that if $\Theta_A^{\mathcal{S}}(0,r) < \varepsilon$ and $\Theta_A^{\mathcal{T}}(0,r) < \tau$, then there exists $s = s(\phi, \Phi, \tau) < 1$ such that $\Theta_A^{\mathcal{T}}(0, sr) < \tau$ as well.

To finish the lemma, assume that $A \subseteq \mathbb{R}^n$, $x \in A$ and $r > 0$ satisfy (2.21) and that

$$\Theta_A^{\mathcal{T}}(x,r) < \eta := s\tau. \tag{2.29}$$

Then by monotonicity and (2.28), it follows that

$$\Theta_A^{\mathcal{T}}(x,tr) < \eta/t \leq \eta/s = \tau \quad \text{for all } s \leq t \leq 1. \tag{2.30}$$

Hence, by induction, it follows that $\Theta_A^{\mathcal{T}}(x,tr) < \tau \Rightarrow \Theta_A^{\mathcal{T}}(x, s^k tr) < \tau$ for all $k \in \mathbb{N}$ and $s \leq t \leq 1$. Hence, $\Theta_A^{\mathcal{T}}(x,r') < \tau$ for all $0 < r' \leq r$. Because $\tau \leq \delta/3 < \delta$, we conclude that $\sup_{0<r'\leq r} \Theta_A^{\mathcal{T}}(x,r') \leq \tau < \delta$. □

2.5 Decompositions of Reifenberg type sets

In this section, we examine decompositions of a set A based on blowup-type.

Theorem 2.5.1 (Decomposition into $\mathcal{T}$ points and $\mathcal{S} \setminus \mathcal{T}$ points)**.** *Let $\mathcal{T} \subset \mathcal{S}$ be local approximation classes such that $\mathcal{T}$ is separated at infinity in $\mathcal{S}$. If $A \subseteq \mathbb{R}^n$ is closed and locally well approximated by $\mathcal{S}$, then A can be written as a disjoint union*

$$A = A_{\mathcal{T}} \cup A_{\mathcal{S}\setminus\mathcal{T}} \quad (A_{\mathcal{T}} \cap A_{\mathcal{S}\setminus\mathcal{T}} = \emptyset) \tag{2.31}$$

such that for every $x \in A_{\mathcal{T}}$, $\mathrm{Tan}(A,x) \subseteq \overline{\mathcal{T}}$ and for every $x \in A_{\mathcal{S}\setminus\mathcal{T}}$, $\mathrm{Tan}(A,x) \subseteq \overline{\mathcal{S}} \setminus \overline{\mathcal{T}}$.

Proof. Assume that $\mathcal{T} \subset \mathcal{S}$ are local approximation classes such that $\mathcal{T}$ is ϕ-separated at infinity in $\mathcal{S}$ for some $\phi > 0$. Suppose that A is closed and locally well approximated by $\mathcal{S}$. Then $\mathrm{Tan}(A,x) \subset \overline{\mathcal{S}}$ for all $x \in A$ by Lemma 2.3.12. Define

$$A_{\mathcal{T}} = \{x \in A : \mathrm{Tan}(A,x) \cap \overline{\mathcal{T}} \neq \emptyset\}$$

and

$$A_{\mathcal{S}\setminus\mathcal{T}} = \{x \in A : \mathrm{Tan}(A,x) \cap \overline{\mathcal{T}} = \emptyset\}.$$

Then $A_{\mathcal{T}} \cap A_{\mathcal{S}\setminus\mathcal{T}} = \emptyset$ and $A = A_{\mathcal{T}} \cup A_{\mathcal{S}\setminus\mathcal{T}} = \emptyset$. On one hand, if $x \in A_{\mathcal{T}}$, then $\mathrm{Tan}(A,x) \subseteq \overline{\mathcal{T}}$, by Theorem 2.4.2. On the other hand, if $x \in A_{\mathcal{S}\setminus\mathcal{T}}$, then we have $\mathrm{Tan}(A,x) \subseteq \overline{\mathcal{S}} \setminus \overline{\mathcal{T}}$. □

We now show that if $\mathcal{T}$ satisfies a stronger assumption than separation at infinity in $\mathcal{S}$, then the set $A_{\mathcal{T}}$ is locally well-approximated by $\mathcal{T}$.

Theorem 2.5.2 (Open/closed decomposition)**.** *Let $\mathcal{T} \subset \mathcal{S}$ be local approximation classes such that $\mathcal{T}$ is (ϕ, Φ)-detectable in $\mathcal{S}$. If $A \subseteq \mathbb{R}^n$ is closed and locally well approximated by $\mathcal{S}$, then A can be written as a disjoint union*

$$A = A_{\mathcal{T}} \cup A_{\mathcal{S}\setminus\mathcal{T}} \quad (A_{\mathcal{T}} \cap A_{\mathcal{S}\setminus\mathcal{T}} = \emptyset) \tag{2.32}$$

with the following properties.

- *$A_{\mathcal{T}}$ is relatively open in A and $A_{\mathcal{T}}$ is locally well approximated by $\mathcal{T}$.*

- *$A_{\mathcal{S}\setminus\mathcal{T}}$ is closed,* $\mathrm{Tan}(A, x) \subseteq \overline{\mathcal{S}} \setminus \overline{\mathcal{T}}$ *for all $x \in A_{\mathcal{S}\setminus\mathcal{T}}$, and for some $\tilde{\eta}(n, \phi, \Phi) > 0$,*

$$\liminf_{r \downarrow 0} \Theta_A^{\mathcal{T}}(x, r) \geq \tilde{\eta} \quad \textit{for all } x \in A_{\mathcal{S}\setminus\mathcal{T}}.$$

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be local approximation classes such that $\mathcal{T} \subseteq \mathcal{S}$ and $\mathcal{T}$ is (ϕ, Φ)-detectable in $\mathcal{S}$. Assume that $A \subseteq \mathbb{R}^n$ is closed and A is locally well approximated by $\mathcal{S}$. Let $\varepsilon = \varepsilon(\phi, \Phi, \delta)$ and $\eta = \eta(\phi, \Phi, \delta)$ be the constants from Lemma 2.4.5 corresponding to $\delta = \phi/2$. Write $\tilde{\eta} = \min\{\eta/8, 1/2\}$. We partition A into two sets as follows. Set

$$A_1 = \left\{x \in A : \liminf_{r\downarrow 0} \Theta_A^{\mathcal{T}}(x, r) < \tilde{\eta}\right\} \quad \text{and} \quad A_2 = \left\{x \in A : \liminf_{r\downarrow 0} \Theta_A^{\mathcal{T}}(x, r) \geq \tilde{\eta}\right\}. \tag{2.33}$$

Then $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Since $\Theta_A^{\mathcal{T}}(x, r_i) \to 0$ along some subsequence $r_i \to 0$ whenever A has a tangent set in $\mathcal{T}$, it is clear that every tangent of A centered at $x \in A_2$ must belong to $\mathcal{S} \setminus \mathcal{T}$. It remains to show that every tangent set of A centered at $x \in A_1$ belongs to $\mathcal{T}$; the set A_1 is relatively open in A; and A_1 is locally well approximated by $\mathcal{T}$.

Fix $x_0 \in A_1$. Because A is locally well approximated by $\mathcal{S}$, there exists $r_0 \in (0, 1)$ such that $\Theta_A^{\mathcal{S}}(x, r) < \varepsilon$ for all $x \in A \cap B(x_0, 1)$ and for all $r \in (0, r_0)$. Since $x_0 \in A_1$, $\liminf_{r\downarrow 0} \Theta_A^{\mathcal{T}}(x_0, r) < \eta/8$. Hence, we can find a scale $r_1 \in (0, r_0/2)$ such that $\Theta_A^{\mathcal{T}}(x_0, 2r_1) < \eta/8$. By Lemma 2.3.5, $\Theta_A^{\mathcal{T}}(x, r_1) < \frac{r_0}{r_1}(2\widetilde{\eta} + (1 + \widetilde{\eta})\eta/8) \leq 2(2\eta/8 + 2\eta/8) = \eta$ for all $x \in A$ such that $|x - x_0| \leq \tilde{\eta}(2r_1)$. Therefore, by Lemma 2.4.5,

$$\Theta_A^{\mathcal{T}}(x, r) < \phi/2 \quad \text{for all } x \in A \cap B(x_0, \tilde{\eta} 2r_1) \text{ and } 0 < r \leq r_1. \tag{2.34}$$

Fix $x \in B(x_0, \tilde{\eta}2r_1)$. We will now show that $\mathrm{Tan}(A,x) \subset \mathcal{T}$. To start, we know that $\mathrm{Tan}(A,x) \subset \mathcal{S}$, because A is locally well approximated by $\mathcal{S}$. Let $S \in \mathrm{Tan}(A,x)$, say $(A-x)/s_i \to S$ in $\mathfrak{C}(0)$ for some sequence $s_i \to 0$. By (2.34), for all i sufficiently large such that $s_i \leq r_1$, we have $\Theta^{\mathcal{T}}_{s_i^{-1}(A-x)}(0,1) < \phi/2$. Hence $\Theta^{\mathcal{T}}_S(0,2/3) \leq (3/4)\phi$. Since $\mathcal{T}$ is (ϕ,Φ)-detectable in $\mathcal{S}$, it follows that $\lim_{t\to 0}\Theta^{\mathcal{T}}_S(0,t) = 0$. Thus, $\mathrm{Tan}(S,0) \subset \mathcal{T}$. In particular, by Lemma 2.3.7(iv) (tangents to tangents are tangents), $\mathrm{Tan}(A,x) \cap \mathcal{T} \neq \emptyset$. Since $\mathrm{Tan}(A,x) \subset \mathcal{S}$, $\mathrm{Tan}(A,x) \cap \mathcal{T} \neq \emptyset$, and $\mathcal{T}$ is separated at infinity from $\mathcal{S}$, we conclude that $\mathrm{Tan}(A,x) \subset \mathcal{T}$ for all $x \in B(x_0,\tilde{\eta})$. Incidentally, this shows that $\liminf_{r\to 0}\Theta^{\mathcal{T}}_A(x,r) < \tilde{\eta}$ for all $x \in B^n(x_0,\tilde{\eta}(2r_1))$. Therefore, every tangent set of A centered at $x \in A_1$ belongs to $\mathcal{T}$ and A_1 is relatively open.

It remains to show that A_1 is locally well approximated by $\mathcal{T}$. Fix any compact set $K \subset A_1$ and $\tau > 0$. Write $\tilde{\tau} = \min\{1/2, \tau/4\}$. Let $\varepsilon = \varepsilon(\phi,\Phi,\delta)$ and $\eta = \eta(\phi,\Phi,\delta)$ be constants from Lemma 2.4.5 corresponding to $\delta = \tau/4$. Since A is locally well approximated by $\mathcal{S}$, there exists $r_0 > 0$ such that $\Theta^{\mathcal{S}}_A(x,r) < \varepsilon$ for all $x \in K$ and $0 < r \leq r_0$. For each $x \in K$, choose $r_x \in (0, r_0/2)$ sufficiently small such that $A \cap B(x,2r_x) \subset A_1$ and $\Theta^{\mathcal{T}}_A(x,r_x) < \eta$. In particular, $A_1 \cap B(x,2r_x) = A \cap B(x,r_2x)$. By Lemma 2.4.5, we have that $\Theta^{\mathcal{T}}_{A_1}(x,r) < \tau/4$ for all $0 < r \leq r_x$. Therefore, $\Theta^{\mathcal{T}}_{A_1}(y,r/2) < \tau$ for all $y \in A_1 \cap B(x,\tilde{\tau}r)$ for all $0 < r \leq r_x$. Finally, since K is compact, it follows that there exists $r_K > 0$ such that for all $x \in K$ and for all $0 < r < r_K$ one has $\Theta^{\mathcal{T}}_{A_1}(x,r) < \tau$. Therefore, A_1 is locally well approximated by $\mathcal{T}$. □

As a corollary to Theorem 2.5.2, we can provide a structure theorem for A near $A_{\mathcal{S}\setminus\mathcal{T}}$ as well. First, we give a definition which is a more flexible quantification of local approximability.

Definition 2.5.3. *Let $A \subseteq \mathbb{R}^n$ be closed, $A' \subseteq A$ be a closed subset, and $\mathcal{S}$ a local approximation class. We say that A is* locally well approximated along A' by $\mathcal{S}$ *if for all compact sets $K \subseteq A'$,*

$$\limsup_{r\downarrow 0}\sup_{x\in A'}\Theta^{\mathcal{S}}_A(x,r) \to 0.$$

Corollary 2.5.4. *Let $\mathcal{T} \subset \mathcal{S}$ be local approximation classes such that $\mathcal{T}$ is (ϕ, Φ)-detectable in $\mathcal{S}$. Set*

$$\mathcal{S}' = \{S \in \overline{\mathcal{S}} : \Theta_S^{\mathcal{T}}(0,r) \geq \phi \text{ for all } r > 0\}.$$

If $A \subseteq \mathbb{R}^n$ is closed and locally well approximated by $\mathcal{S}$, then A is locally well approximated along $A_{\mathcal{S}\setminus\mathcal{T}}$ by $\mathcal{S}'$; that is, for all compact sets $K \subseteq A_{\mathcal{S}\setminus\mathcal{T}}$, $\lim_{r\downarrow 0} \sup_{x\in K} \Theta_A^{\mathcal{S}'}(x,r) = 0$.

Proof. Suppose, for contradiction, that $K \subseteq A_{\mathcal{S}\setminus\mathcal{T}}$ is a compact set such that

$$\limsup_{r\downarrow 0} \sup_{x\in K} \Theta_A^{\mathcal{S}'}(x,r) > 0.$$

Then there is a sequence $x_i \in K$ and a sequence of radii $r_i \downarrow 0$ such that $\Theta_A^{\mathcal{S}'}(x_i,r_i) > c_0 > 0$ for all i. Because K is compact, by passing to a subsequence, we can assume that $x_i \to x \in K$ and $(A - x_i)/r_i \to S$ for some $S \in \overline{\mathcal{S}}$. For convenience, set $A_i = (A - x_i)/r_i$. Note that by scale invariance, $\Theta_{A_i}^{\mathcal{S}'}(0,1) = \Theta_A^{\mathcal{S}'}(x_i,r_i) > c_0$. Because $A_i \to S$, Lemma 2.3.4 implies that $S \notin \overline{\mathcal{S}'}$. By definition, there exists a radius $\rho > 0$ such that $\Theta_S^{\mathcal{T}}(0,\rho) < \phi$. Fix such a ρ. Let $\tilde{\eta} = \tilde{\eta}(n,\phi,\Phi) > 0$ be the constant from Lemma 2.5.2. Let $\delta = \tilde{\eta}/2$, and let $\eta = \eta(\phi,\Phi,\delta)$ and $\varepsilon = \varepsilon(\phi,\Phi,\delta)$ be the constants from Lemma 2.4.5.

By detectability, we have that there is an $s > 0$ small enough such that $\Theta_S^{\mathcal{T}}(0,s\rho) \leq \Phi(s) < \eta/8$. Set $\lambda = s\rho/2$, so that $2\lambda = s\rho$. Thus there is a $T \in \mathcal{T}$ such that $\widetilde{\mathrm{D}}^{0,2\lambda}[S,T] < \eta/8$. By the quasitriangle inequality, we obtain $\widetilde{\mathrm{D}}^{0,\lambda}[A_i,T] \leq 2(\widetilde{\mathrm{D}}^{0,2\lambda}[A_i,S] + \widetilde{\mathrm{D}}^{0,2\lambda}[S,T]) \leq 2\,\widetilde{\mathrm{D}}^{0,2\lambda}[A_i,S] + \eta/4$. Because $A_i \to S$, for all i large enough, $\widetilde{\mathrm{D}}^{0,2\lambda}[A_i,S] < \eta/8$. Thus, for all i large enough, $\widetilde{\mathrm{D}}^{x_i,\lambda r_i}[A, r_iT + x_i] = \widetilde{\mathrm{D}}^{0,\lambda}[A_i,T] < \eta/2$. In particular $\Theta_A^{\mathcal{T}}(x_i,r_i\lambda) < \eta/2$, for i large enough.

We are now ready to derive a contradiction. Because A is locally well approximated by $\mathcal{S}$, there is an r_0 small enough such that for all $x \in K$, $\sup_{0<r\leq r_0} \Theta_A^{\mathcal{S}}(x,r) < \varepsilon$. In particular, for i large enough such that $\lambda r_i \leq r_0$, we have that $\sup_{0<r\leq\lambda r_i} \Theta_A^{\mathcal{S}}(x_i,r) < \varepsilon$. Combining this with the fact that $\Theta_A^{\mathcal{T}}(x_i,r_i\lambda) < \eta/2$ for i large enough, we apply Lemma 2.4.5 to get that $\sup_{0<r\leq\lambda r_i} \Theta_A^{\mathcal{T}}(x_i,r) < \delta = \tilde{\eta}/2 < \tilde{\eta}$. However, this contradicts the second bullet of Theorem 2.5.2, which is that $\liminf_{r\downarrow 0} \Theta_A^{\mathcal{T}}(x,r) \geq \tilde{\eta}$ for all $x \in A_{\mathcal{S}\setminus\mathcal{T}}$. $\square$

Chapter 3

UNILATERAL LOCAL SET APPROXIMATION

3.1 Mattila-Vuorinen type sets and Minkowski dimension

We now investigate unilateral local set approximation, or local set approximation where we require that A be close in small balls $B(x,r)$ to a given class of approximants $S_{x,r}$, but we do not ask that $S_{x,r}$ be close to A. The Jones beta number in $B(x,r)$ is defined as

$$\beta_\infty(x,r) = \inf_{P \ni x} \widetilde{\mathrm{d}}^{\,x,r}(A,P)$$

where P ranges over all planes containing x [16]. Thus, asking that β_∞ be small gives a different notion of flatness than does Reifeberg flatness. Intuitively, it is a flatness which allows for "holes." Mattila and Vuorinen [26] similarly say that a set $A \subset \mathbb{R}^n$ has the (m,ε)-*linear approximation property* if for each $x \in A$ and $r > 0$ there exists an m-dimensional plane $V \in \mathcal{G}(n,m)$ such that $A \cap B(x,r) \subset \{y \in \mathbb{R}^n : \mathrm{dist}(y,V) \le \varepsilon r\}$ (see Example 2.1). The following definition is a natural generalization of the Jones beta number to an arbitrary local approximation class.

Definition 3.1.1 (Mattila-Vuorinen type sets)**.** *Let $A \subseteq \mathbb{R}^n$, $x \in A$, $r > 0$, and let $\mathcal{S}$ be a local approximation class.*

i. Define the unilateral approximability of A by $\mathcal{S}$ in $B(x,r)$ *by*

$$\beta_A^{\mathcal{S}}(x,r) = \inf_{S \in \mathcal{S}} \widetilde{\mathrm{d}}^{\,x,r}(A, S+x)$$

(see Section 2.2 for the definition of $\widetilde{\mathrm{d}}^{\,x,r}$).

ii. For $\delta > 0$, we say that A is locally unilaterally δ-approximated by $\mathcal{S}$ *if for all compact sets $K \subseteq \mathbb{R}^n$, there exists r_K such that for all $0 < r \le r_K$ and $x \in K \cap A$, $\beta_A^{\mathcal{S}}(x,r) \le \delta$.*

iii. We say that A is locally unilaterally well approximated by $\mathcal{S}$ *if it is locally δ-unilaterally approximable by $\mathcal{S}$ for all $\delta > 0$.*

Remark 3.1.2. *Note that in the definition of bilateral local approximability, we ask that $\Theta_A^{\mathcal{S}}$ be small locally uniformly, in the sense of compact subsets of A, whereas in the definition of a locally unilaterally approximable set, we ask that $\beta_A^{\mathcal{S}}$ be small locally uniformly in the sense of compact subsets of $\mathbb{R}^n$. The choice of an intrinsic local quantification or an extrinsic local quantification can drastically affect the utility of a definition. For locally approximable sets, we are often interested in decomposing sets, making intrinsic locality the right quantification. For locally unilaterally approximable sets, we focus in this paper on Minkowski-dimension bounds for which one needs extrinsic quantification. Intuitively, we would like to apply the technology of unilateral approximability to sets which are full of holes, and this requires extrinsic quantification to provide Minkowski content estimates.*

It immediately follows from the definitions that every subset of a Reifenberg type set is a Matilla-Vuorinen type set. That is, if A is locally ε-approximable by $\mathcal{S}$ and $B \subseteq A$, then B is locally ε-unilaterally approximable by $\mathcal{S}$. The converse fails.

Example. [David-Toro [13], Example 12.4] For all $0 < \varepsilon \ll 1$, there exists a Möbius strip M in $\mathbb{R}^3$ which has the $(2, \varepsilon)$-linear approximation property, but which cannot be included in a δ-Reifenberg flat at large scales for any small δ. By removing small balls from M and replacing them with tiny copies of M in the appropriate way, one can obtain a set which has the $(2, \varepsilon)$-linear approximation property but which is not included in a δ-Reifenberg flat set at any scale.

The defining property of Mattila-Vuorinen type sets interacts naturally with the notion of Minkowski dimension.

Definition 3.1.3 (Minkowski dimension via covering numbers)**.** *For all nonempty bounded sets $A \subset \mathbb{R}^n$ and $0 < r < \infty$, define the covering number*

$$\mathrm{N}(A, r) = \min \left\{ k : A \subseteq \bigcup_{i=1}^{k} B(x_i, r) \text{ for some } x_i \in \mathbb{R}^n \right\}.$$

The lower *and* upper Minkowski dimension *of A are given by*

$$\underline{\dim}_M(A) = \liminf_{r \downarrow 0} \frac{\log(\mathrm{N}(A, r))}{\log(1/r)} \quad \text{and} \quad \overline{\dim}_M(A) = \limsup_{r \downarrow 0} \frac{\log(\mathrm{N}(A, r))}{\log(1/r)},$$

respectively. If A is a nonempty unbounded set, we define the lower *and* upper Minkowski dimension *of A by*

$$\underline{\dim}_M(A) = \lim_{R\to\infty} \underline{\dim}_M(A \cap B(0,R)) \quad \textit{and} \quad \overline{\dim}_M(A) = \lim_{R\to\infty} \overline{\dim}_M(A \cap B(0,R)).$$

Note that the limits above are well defined because both the lower and upper Minkowski dimensions will be nondecreasing and bounded in R.

In order to control Minkowski dimension by the quantity $\beta_A^{\mathcal{S}}$, we will need to assume a uniform control on the "local" covering numbers, which we here develop.

Definition 3.1.4 (Coverings and Minkowski Profiles)**.** *Let $A \subseteq \mathbb{R}^n$, $x \in A$, $r > 0$, and let $\mathcal{S}$ be a local approximation class.*

*i. For $\varepsilon > 0$, an ε-*covering of *A is a collection $\mathcal{B}$ of balls centered in A with radius ε such that $A \subseteq \cup\mathcal{B}$.*

*ii. The ε-*covering number of *A* in *$B(x,r)$ is defined as*

$$\mathrm{N}^{x,r}(A,\varepsilon) = \inf\{|\mathcal{B}| : \mathcal{B} \textit{ is an } \varepsilon\textit{-covering of } A \cap B(x,r)\}.$$

*iii. For $\alpha, C > 0$ and $\varepsilon \in (0,1]$, we say that $\mathcal{S}$ has a (C,α,ε)-*Minkowski profile *if for all $R > 0$, $s \in (0,\varepsilon]$, and $S \in \mathcal{S}$,*

$$\mathrm{N}^{x,R}(S, sR) \leq C/s^\alpha.$$

Theorem 3.1.5. *Let $\mathcal{S}$ be a local approximation class with $(C_{\mathcal{S}},\alpha,\varepsilon)$-Minkowski profile. Let δ and Λ be chosen such that $\delta \in (0, \min(1,2\varepsilon)/2\Lambda)$ and $\Lambda = 1 + C_{\mathcal{S}}^{1/\alpha}(1+\delta)$. If $A \subseteq \mathbb{R}^n$ is locally δ-unilaterally approximable by $\mathcal{S}$, then*

$$\overline{\dim}_M(A) \leq \frac{\alpha}{1 - \frac{\log(2\Lambda)}{\log(1/\delta)}}. \tag{3.1}$$

Remark 3.1.6. *Note that for all $\delta \in (0, \min(1,2\varepsilon)/(2 + 4C_{\mathcal{S}}^{1/\alpha})]$, $\delta \in (0, \min(1,2\varepsilon)/2\Lambda)$, and so the condition is satisfied on some nonempty open interval in δ.*

We first prove the following lemma which we will use as the quantitative foundation of the proof of Theorem 3.1.5.

Lemma 3.1.7. *Let $\mathcal{S}$ be a local approximation class with $(C_{\mathcal{S}}, \alpha, \varepsilon)$-Minkowski profile. Set $\Lambda = 1 + C_{\mathcal{S}}^{1/\alpha}(1+\delta)$, and let $\delta \in (0, \varepsilon/\Lambda)$. Assume that $A \subseteq \mathbb{R}^n$, $0 \in A$, and $\beta_A^{\mathcal{S}}(0,1) < \delta$. Then $\mathrm{N}^{0,1}(A, 2\Lambda\delta) \le 1/\delta^{\alpha}$.*

Proof. By assumption, there exists $S \in \mathcal{S}$ such that $\widetilde{\mathrm{d}}^{\,0,1}(A, S) < \delta$. Let $\mathcal{B}$ be an optimal $C_{\mathcal{S}}^{1/\alpha}\delta(1+\delta)$-cover of $S \cap B(0, 1+\delta)$. Note that $C_{\mathcal{S}}^{1/\alpha}\delta < C_{\mathcal{S}}^{1/\alpha}\varepsilon/\Lambda < \varepsilon$. Because $\mathcal{S}$ has a $(C_{\mathcal{S}}, \alpha, \varepsilon)$-Minkowski profile, we get that

$$|\mathcal{B}| = \mathrm{N}^{0,1+\delta}(S, C_{\mathcal{S}}^{1/\alpha}\delta(1+\delta)) \le C_{\mathcal{S}}/(C_{\mathcal{S}}^{1/\alpha}\delta)^{\alpha} = 1/\delta^{\alpha}.$$

Enumerate the centers of the balls in $\mathcal{B}$ as x_i for $i = 1, \dots, N$.

We now claim that the balls $B(x_i, \Lambda\delta)$ cover A. Let $y \in A \cap B(0,1)$. Then there exists some $x \in S \cap B(0, 1+\delta)$ such that $|x - y| < \delta$. But $S \cap B(0, 1+\delta) \subseteq \cup\mathcal{B}$. Thus, there is some x_i such that $|x - x_i| < C_{\mathcal{S}}^{1/\alpha}(1+\delta)\delta$. By the triangle inequality, we have that $|y - x_i| < (1 + C_{\mathcal{S}}^{1/\alpha}(1+\delta))\delta = \Lambda\delta$, establishing the claim.

Let I be the set of indices i such that $B(x_i, C_{\mathcal{S}}^{1/\alpha}(1+\delta)\delta) \cap A \neq \emptyset$. For each $i \in I$, let $y_i \in B(x_i, (1+C_{\mathcal{S}}^{1/\alpha}(1+\delta))\delta) \cap A$. Then we have that $\mathcal{B}' = \{B(y_i, 2\Lambda\delta) : i \in I\}$ is a covering of $A \cap B(0,1)$ of balls centered in A with size $|\mathcal{B}'| \le 1/\delta^{\alpha}$. □

Proof of Theorem 3.1.5. Let $A \subseteq \mathbb{R}^n$ be locally δ-unilaterally approximable by $\mathcal{S}$. Let $R > 0$ be large enough such that $A \cap B(0,R) \neq \emptyset$. Let $\delta' \in (\delta, \min(1, 2\varepsilon)/2\Lambda)$. Let r_R be small enough such that for $x \in A \cap B(0,R)$ and $0 < r \le r_R$, $\beta_A^{\mathcal{S}}(x,r) < \delta'$. Let $\Lambda' = 1 + C_{\mathcal{S}}^{1/\alpha}(1+\delta')$. Note that $2\Lambda > \Lambda' > \Lambda$.

Note that $\{B(x, r_R) : x \in A\}$ covers $\overline{A \cap B(0,R)}$, which is compact. Thus, we take a finite subcover with centers x_i for $i = 1, \dots, M$. We now prove that for each i,

$$\overline{\dim}_M(A \cap B(x_i, r_R)) \le \frac{\alpha}{1 - \frac{\log(2\Lambda')}{\log(1/\delta')}}. \tag{3.2}$$

See equation (3.1) for comparison. Note that once we have established (3.2), we have that $\overline{\dim}_M(A \cap B(0,R))$ satisfies the same bound because Minkowski dimension is preserved by finite unions. Because $R > 0$ was arbitrary, we thus get the same bound for A. Noting that this bound holds for all δ' in some small window above δ, we get that equation (3.1) holds. Thus, we have only to establish (3.2).

Fix any $i \in \{1, \ldots, M\}$ and set $A' = (A - x_i)/r_R \cap B(0,1)$. Set $s = 2\Lambda\delta'$. Note that $s < 1$ by assumption. We now claim that

$$\mathrm{N}^{0,1}(A', s^k) < 1/(\delta')^{k\alpha} \quad \text{for} \quad k = 1, 2, \ldots \tag{3.3}$$

We prove (3.3) by induction. The base case, $k = 1$, follows from Lemma 3.1.7 with A taken to be A' and δ taken to be δ'. Suppose now that $\mathrm{N}^{0,1}(A', s^k) < 1/(\delta')^{k\alpha}$. Let $\mathcal{B}^{(k)} = \{B(x_j, s^k)\}_{j=1}^N$ be an optimal s^k-covering of A'. By assumption, $N \leq 1/(\delta')^{k\alpha}$. Note that for each j, $A'' = (A' - x_j)/s^k$ satisfies the assumptions of Lemma 3.1.7 with δ taken to be δ' and A taken to be A''. Thus this yields an s-covering $\{B(x_{j,\ell}, s)\}_\ell$ of $A'' \cap B(0,1)$ of size at most $1/\delta^\alpha$. By doing this for each j, we define an s^{k+1}-covering of A'

$$\mathcal{B}^{(k+1)} = \{B(x_{j,\ell} + x_j, s^{k+1})\}_{j,\ell}$$

such that $|\mathcal{B}^{(k+1)}| \leq 1/\delta^{(k+1)\alpha}$. Thus, we have established (3.3).

Next we observe that $\overline{\dim}_M(A') < \alpha/(1 - \log(2\Lambda')/\log(1/\delta'))$. To see this, we note that

$$\frac{\log(\mathrm{N}(A', s^k))}{\log(s^{-k})} \leq \frac{\log(\delta'^{-k\alpha})}{\log((2\lambda\delta')^{-k})} = \frac{-k\alpha\log(\delta')}{-k(\log(2\Lambda') + \log(\delta'))} = \frac{\alpha}{1 - \frac{\log(2\Lambda')}{\log(1/\delta')}}.$$

We now have the appropriate bound at a geometric sequence of scales s^k. It is a standard argument that this suffices to bound Minkowski dimension, but we give it here for completion. It is not hard to see that for any $r \in [s^k, s^{k-1}]$, we have that

$$\frac{\log(\mathrm{N}(A', r))}{\log(r)} \leq \frac{\log(\delta'^{-k\alpha})}{\log(s(2\lambda\delta')^{-k})} = \frac{-k\alpha\log(\delta')}{-k(\log(2\Lambda') + \log(\delta')) + \log(s)} \to \frac{\alpha}{1 - \frac{\log(2\Lambda')}{\log(1/\delta')}}.$$

Hence, we get that $\overline{\dim}_M(A') < \alpha/(1 - \log(2\Lambda')/\log(1/\delta'))$. As observed earlier, this completes the proof of Theorem 3.1.5.

□

Remark 3.1.8. *We note that Theorem 3.1.5 is best suited for constants $C_{\mathcal{S}} \gg 1$. For constants $C_S < 1$, the bounds of Theorem 3.1.5 are not in general tight. Though not quite* *in this language, Salli showed in [32] that sets which are locally δ-unilaterally approximated by centered half-spaces have dimension bounded by $n-1+C/\log(1/\delta)$, whereas our estimate would only give a completely trivial bound of $n+C/\log(1/\delta)$. The further improvement on dimension bounds in the case $C_{\mathcal{S}}<1$ is also highly dependent on ε in this case.*

Corollary 3.1.9. *Let $\mathcal{S}$ be a local approximation class with $(C_{\mathcal{S}},\alpha,\varepsilon)$-Minkowski profile for some constants $C_{\mathcal{S}}$ and $\varepsilon>0$. If $A\subseteq\mathbb{R}^n$ is locally unilaterally well approximated by $\mathcal{S}$, then $\overline{\dim}_M(A)\le\alpha$.*

3.2 Upper pseudotangents and unilateral approximability

We here formulate some theory for unilateral approximations analogous to the theory developed for bilateral approximations. We then give an application of the technology to Minkowski dimension bounds on the singular sets of some sets.

Definition 3.2.1. *Let $A\subseteq\mathbb{R}^n$, not necessarily closed, and let $\mathcal{S}$ be a local approximation class. We say that $\mathcal{S}$ is an* upper tangent class at $x\in\overline{A}$ *if for each $T\in\operatorname{Tan}(\overline{A},x)$, there is an $S\in\mathcal{S}$ with $T\subseteq S$. We say that $\mathcal{S}$ is an* upper pseudotangent class at $x\in\overline{A}$ *if for each $T\in\Psi\text{-}\operatorname{Tan}(\overline{A},x)$, there is an $S\in\mathcal{S}$ with $T\subseteq\mathcal{S}$.*

Lemma 3.2.2. *Let $\mathcal{S}$ be a local approximation class, $A\subseteq\mathbb{R}^n$, and $x\in\overline{A}$. Then we have that $\lim_{r\downarrow 0}\beta_A^{\mathcal{S}}(x,r)=0$ if and only if $\overline{\mathcal{S}}$ is an upper tangent class as x for A.*

Proof. Without loss of generality, assume $x=0$. Suppose that $\overline{\mathcal{S}}$ is an upper tangent class of 0. Let $r_i\downarrow 0$. By sequential compactness, there exists a subsequence r_{ij} such that $A/r_{ij}\to T$ for some $T\in\operatorname{Tan}(A,0)$. Thus, $\widetilde{\mathrm{D}}^{0,1}[A/r_{ij},T]\to 0$. By assumption, there exists $S\in\overline{\mathcal{S}}$ such that $T\subseteq S$. By the scale invariance of $\widetilde{\mathrm{D}}$ (see (2.3)), we have that $\widetilde{\mathrm{D}}^{0,1}[A/r_{ij},T]=\widetilde{\mathrm{D}}^{0,r_{ij}}[A,r_{ij}T]$. Because $\mathcal{S}$ is a local approximation class, $r_{ij}T\subseteq r_{ij}S\in\overline{\mathcal{S}}$ and so $\beta_A^{\mathcal{S}}(0,r_{ij})\le\widetilde{\mathrm{D}}^{0,r_{ij}}[A,r_{ij}T]\to 0$. Hence, we conclude that $\lim_{r\downarrow 0}\beta_A^{\mathcal{S}}(0,r)=0$.

Conversely, suppose that $\lim_{r\downarrow 0}\beta_A^{\mathcal{S}}(0,r)=0$. Suppose that $T\in\operatorname{Tan}(A,0)$. Then there exists a sequence $r_i\downarrow 0$ such that $A/r_i\to T$. Fix $R>0$. Because $A/r_i\to T$, we have that $\widetilde{\mathrm{D}}^{0,R}[A/r_i,T]\to 0$. On the other hand, because $\lim_{r\downarrow 0}\beta_A^{\mathcal{S}}(0,r)=0$, we have that there exists a sequence $S(i,R)\in\mathcal{S}$ such that $\widetilde{\mathrm{d}}^{\,0,Rr_i}[A,S(i,R)]\to 0$ as $i\to\infty$. By the scale invariance of $\widetilde{\mathrm{d}}^{x,r}$, we have that $\widetilde{\mathrm{d}}^{0,R}[A/r_i, S(i,R)/r_i] = \widetilde{\mathrm{d}}^{0,Rr_i}[A, S(i,R)] \to 0$. Let $S'(i,R) = S(i,R)/r_i$. By the quasitriangle inequality, we have that $\widetilde{\mathrm{d}}^{0,R/2}[T, S'(i,R)] \leq 2(\widetilde{\mathrm{d}}^{0,R}[T, A/r_i] + \widetilde{\mathrm{d}}^{0,R}[A/r_i, S'(i,R)]) \to 0$ for any $R > 0$. Thus we can choose a subsequence $(i_k) \subseteq (i)$ such that $\widetilde{\mathrm{d}}^{0,k}[S'(i_k, 2k), T] \leq 1/k^2$. By monotonicity, for any radius R and $k \geq R$, we have that $\widetilde{\mathrm{d}}^{0,R}[T, S'(i_k, 2k)] \leq k/R\, \widetilde{\mathrm{d}}^{0,k}[T, S'(i_k, 2k)] \leq k/R \cdot 1/k^2 = 1/(Rk) \to 0$. By taking a further subsequence $(k') \subseteq (k)$, we have that $S'(i_{k'}, 2k') \to S$ for some $S \in \overline{\mathcal{S}}$. With the previous inequalities, this shows that $T \subseteq S$. Thus we have that $T \subseteq S \in \overline{\mathcal{S}}$. We conclude that $\overline{\mathcal{S}}$ is an upper tangent class at 0. □

Lemma 3.2.3. *Let $\mathcal{S}$ be a local approximation class, let $A \subseteq \mathbb{R}^n$ be a nonempty set. Then A is locally unilaterally well approximated by $\mathcal{S}$ if and only if $\overline{\mathcal{S}}$ is an upper pseudotangent class at x for all $x \in \overline{A}$.*

Proof. Suppose that $\overline{\mathcal{S}}$ is an upper pseudotangent class at x for all $x \in \overline{A}$. Let $K \subseteq \overline{A}$ be a compact set. Set $\ell = \limsup_{r \downarrow 0} \sup_{x \in K} \beta_A^{\mathcal{S}}(x,r)$. Then there exist sequences $x_i \in K$, $r_i \downarrow 0$ such that $\beta_A^{\mathcal{S}}(x_i, r_i) \to \ell$. Because K is compact, there is a subsequence $(i_j) \subseteq (i)$ such that $x_{ij} \to x$ for some $x \in K$. By sequential compactness of $\mathfrak{C}(0)$, there is a subsequence $(i_k) \subseteq (i_j)$ such that $(A - x_{ik})/r_{ik} \to T$ for some $T \in \Psi\text{-Tan}(A, x)$. By assumption, $T \subseteq S \in \overline{\mathcal{S}}$. Thus, $\beta_A^{\mathcal{S}}(x_i, r_i) \leq \widetilde{\mathrm{d}}^{x_i,r_i}[A, r_i S + x_i] \leq \widetilde{\mathrm{d}}^{x_i,r_i}[A, r_i T + x_i] = \widetilde{\mathrm{d}}^{0,1}[(A - x_i)/r_i, T] \to 0$. Hence, $\ell = 0$. Because $K \subseteq A$ was an arbitrary compact set, we establish that $\Theta_A^{\mathcal{S}}(x,r) \to 0$ uniformly on compact subsets of A. So by definition, A is locally unilaterally well approximated by $\mathcal{S}$.

Suppose that A is locally unilaterally well approximated by $\mathcal{S}$. Let $x \in \overline{A}$ and $T \in \Psi\text{-Tan}(\overline{A}, x)$. Then there exist sequences $x_i \to x$ and $r_i \downarrow 0$ such that $(A - x_i)/r_i \to T$. Fix $R > 0$. Because $(\overline{A} - x_i)/r_i \to T$, we have that $\widetilde{\mathrm{D}}^{0,R}[(A - x_i)/r_i, T] \to 0$. Let $X := \{x_i\} \cup \{x\}$. Then X is compact, so by assumption $\lim_{r \downarrow 0} \sup_{y \in X} \beta_A^{\mathcal{S}}(y, r) = 0$. Thus, we have that there exists a sequence $S(i,R) \in \mathcal{S}$ such that $\widetilde{\mathrm{d}}^{x_i,Rr_i}[A, S(i,R) + x_i] \to 0$ as $i \to \infty$. By the scaling of $\widetilde{\mathrm{D}}$, we have that $\widetilde{\mathrm{D}}^{0,R}[(A - x_i)/r_i, S(i,R)/r_i] = \widetilde{\mathrm{D}}^{0,Rr_i}[A, S(i,R)] \to 0$. Let $S'(i,R) = S(i,R)/r_i$. By the quasitriangle inequality, we have that $\widetilde{\mathrm{d}}^{0,R/2}[T, S'(i,R)] \leq 2(\widetilde{\mathrm{d}}^{0,R}[T, (A - x_i)/r_i] + \widetilde{\mathrm{D}}^{0,R}[(A - x_i)/r_i, S'(i,R)]) \to 0$ for any $R > 0$. Thus we can choose a subsequence $(i_k) \subseteq (i)$ such that $\widetilde{\mathrm{d}}^{0,k}[T, S'(i_k, 2k)] \leq 1/k^2$. By monotonicity, for any radius R and $k \geq R$, we have that $\widetilde{\mathrm{d}}^{\,0,R}[T, S'(i_k, 2k)] \leq k/R\,\widetilde{\mathrm{d}}^{\,0,k}[T, S'(i_k, 2k)] \leq k/R \cdot 1/k^2 = 1/(Rk) \to 0$. Further, by compactness of $\mathfrak{C}(0)$ (Theorem 2.2.4), we can choose a subsequence $(k') \subseteq (k)$ such that $S'(i_{k'}, 2k') \to S \in \overline{\mathcal{S}}$. This implies that the sequence $T \subseteq S$. Because T was an arbitrary pseudotangent at an arbitrary point, we conclude that $\overline{\mathcal{S}}$ is everywhere an upper pseudotagent class of A. □

3.3 Dimension bounds on the singular set

We now give a method of providing Minkowski dimension bounds on the "singular" part of a closed set $A \subseteq \mathbb{R}^n$. We will do this by considering local approximation classes $\mathcal{T} \subseteq \mathcal{S}$ with $\mathcal{T}$ (ϕ, Φ)-detectable in $\mathcal{S}$ such that A is locally well approximated by $\mathcal{S}$ (in the bilateral sense). Recall that by Theorem 2.5.1 we can write $A = A_{\mathcal{T}} \cup A_{\mathcal{S}\setminus\mathcal{T}}$ where

$$A_{\mathcal{T}} = \{x \in A : \mathrm{Tan}(A, x) \subseteq \overline{\mathcal{T}}\} \quad \text{and} \quad A_{\mathcal{S}\setminus\mathcal{T}} = \{x \in A : \mathrm{Tan}(A, x) \subseteq \overline{\mathcal{S}} \setminus \overline{\mathcal{T}}\}.$$

Stated more precisely, we seek to prove that $\overline{\dim}(A_{\mathcal{S}\setminus\mathcal{T}}) \leq \alpha$ if the "singular" class of $\mathcal{S}$ with respect to $\mathcal{T}$ (see Definition 3.3.1) has an α-Minkowski profile.

We begin by recalling the relevant facts of Chapter 2. By $\mathfrak{C}(x)$ we denote the collection of closed sets passing through x. We recall that because $\mathcal{T}$ is detectable in $\mathcal{S}$, by Theorem 2.5.2, $A_{\mathcal{T}}$ is relatively open and locally well approximated by $\mathcal{T}$, whereas $A_{\mathcal{S}\setminus\mathcal{T}}$ is closed. Additionally, by Corollary 2.5.4, we have that if we set

$$\mathcal{S}' = \{X \in \overline{\mathcal{S}} : \Theta_X^{\mathcal{T}}(0, r) \geq \phi \text{ for all } r > 0\}, \tag{3.4}$$

then A is locally well approximated by $\mathcal{S}'$ along $A_{\mathcal{S}\setminus\mathcal{T}}$ (see Definition 2.5.3). In order to avoid notational confusion, in this section we label our elements of $\overline{\mathcal{S}}$ by $X \in \overline{\mathcal{S}}$ (rather than labeling our sets S).

We will also use the following fundamental observation: if $X \in \overline{\mathcal{S}}$, then X is (globally) well approximated by $\mathcal{S}$, and so we can decompose $X = X_{\mathcal{T}} \cup X_{\mathcal{S}\setminus\mathcal{T}}$. Moreover, we observe that $X_{\mathcal{S}\setminus\mathcal{T}} = \{x \in X : \Theta_X^{\mathcal{T}}(x, r) \geq \phi \text{ for all } r > 0\}$ as follows. By detectability, if $\Theta_X^{\mathcal{T}}(x, r) < \phi$, then for all $s \in (0, 1)$, $\Theta_X^{\mathcal{T}}(x, sr) < \Phi(s) \to 0$ as $s \to 0$. Thus we conclude that $x \in X_{\mathcal{T}}$, and the observation follows.

Definition 3.3.1. *Let $\mathcal{T} \subseteq \mathcal{S}$ be local approximation classes. We define the $\mathcal{T}$* singular class of *$\mathcal{S}$ to be*

$$\operatorname{sing}_{\mathcal{T}}(\mathcal{S}) = \{\Sigma \in \mathfrak{C}(0) : \Sigma = X_{\mathcal{S}\setminus\mathcal{T}} \text{ for some } X \in \overline{\mathcal{S}}\}.$$

By the previous paragraph, we record the observation that using the notation (3.4),

$$\operatorname{sing}_{\mathcal{T}}(\mathcal{S}) = \{X_{\mathcal{S}\setminus\mathcal{T}} : X \in \mathcal{S}'\}. \tag{3.5}$$

This will be the key observation in the following theorem.

Theorem 3.3.2. *Let $\mathcal{T} \subseteq \mathcal{S}$ be local approximation classes such that $\mathcal{T}$ is detectable in $\mathcal{S}$. Assume that $A \subseteq \mathbb{R}^n$ is closed and locally well approximated by $\mathcal{S}$. Then $A_{\mathcal{S}\setminus\mathcal{T}}$ is locally unilaterally well approximated by* $\operatorname{sing}_{\mathcal{T}}(\mathcal{S})$.

Proof. We prove this theorem by showing that $\operatorname{sing}_{\mathcal{T}}(\mathcal{S})$ is an upper pseudotangent class for $A_{\mathcal{S}\setminus\mathcal{T}}$. Recall that $A_{\mathcal{S}\setminus\mathcal{T}}$ is closed.

Let $x \in A_{\mathcal{S}\setminus\mathcal{T}}$, and $\Sigma \in \Psi\text{-Tan}(A_{\mathcal{S}\setminus\mathcal{T}}, x)$. We will show that there exists an $X \in \Psi\text{-Tan}(A, x)$ such that $\Sigma \subseteq X_{\mathcal{S}\setminus\mathcal{T}}$. By definition, there exist sequences $x_i \in A_{\mathcal{S}\setminus\mathcal{T}}$, $x_i \to x$, and $r_i \downarrow 0$ such that $(A_{\mathcal{S}\setminus\mathcal{T}} - x_i)/r_i \to \Sigma$. We choose a subsequence (without relabeling) such that $(A - x_i)/r_i \to X \in \Psi\text{-Tan}(A, x) \subseteq \overline{\mathcal{S}}$.

Claim. $\Sigma \subseteq X_{\mathcal{S}\setminus\mathcal{T}}$.

Suppose, for contradiction, that $y \in \Sigma \setminus X_{\mathcal{S}\setminus\mathcal{T}}$. Because $y \in \Sigma$, there exists a sequence $y_i \in (A_{\mathcal{S}\setminus\mathcal{T}} - x_i)/r_i$ such that $y_i \to y$. Write $y_i = (z_i - x_i)/r_i$ for $z_i \in A_{\mathcal{S}\setminus\mathcal{T}}$. It follows that

$$\frac{A - z_i}{r_i} = \frac{A - x_i}{r_i} + \frac{x_i - z_i}{r_i} \to X - y \tag{3.6}$$

(To see this, one can apply monotonicity to the ball $B(0, R + |y|)$ to get bounds in the ball $B(y, R)$.) Because $z_i \in A_{\mathcal{S}\setminus\mathcal{T}}$, we can apply Corollary 2.5.4 to get that for any $R > 0$, $\Theta^{\mathcal{S}'}_A(z_i, Rr_i) \to 0$ for all $R > 0$. Hence, setting $A_i = (A - z_i)/r_i$, we get that $\Theta^{\mathcal{S}'}_{A_i}(0, R) \to 0$ for all $R > 0$. Thus $X - y \in \mathcal{S}'$. In particular, $\Theta^{\mathcal{T}}_{(X-y)}(0, r) \geq \phi$ for all $r > 0$, and we get that $0 \in (X - y)_{\mathcal{S}\setminus\mathcal{T}}$. Thus $y \in X_{\mathcal{S}\setminus\mathcal{T}}$. Because we assumed that $y \notin X_{\mathcal{S}\setminus\mathcal{T}}$, we have derived a contradiction and proven the claim.

By noting that $0 \in (A_{\mathcal{S}\setminus\mathcal{T}} - x_i)/r_i$ for all i, $0 \in \Sigma$. In particular, $0 \in X_{\mathcal{S}\setminus\mathcal{T}} \subseteq X$. Thus, we conclude that $\Sigma \subseteq X_{\mathcal{S}\setminus\mathcal{T}}$ and $X \in \mathcal{S}'$, proving the theorem.

□

Corollary 3.3.3. *Suppose that $\mathcal{S}$ is a local approximation class and $\mathcal{T} \subseteq \mathcal{S}$ is detectable. Let $C, \alpha, \varepsilon > 0$ and assume that $\operatorname{sing}_{\mathcal{T}}(\mathcal{S})$ admits a (C, α, ε)-Minkowski profile. Then for any set A which is locally well approximated by $\mathcal{S}$, $\overline{\dim}(A_{\mathcal{S}\setminus\mathcal{T}}) \leq \alpha$.*

Proof. Because $\mathcal{T}$ is detectable in $\mathcal{S}$, Theorem 3.3.2 implies that we have that $\operatorname{sing}_{\mathcal{T}}(\mathcal{S})$ is an upper pseudotangent class of $A_{\mathcal{S}\setminus\mathcal{T}}$. Thus, $A_{\mathcal{S}\setminus\mathcal{T}}$ is locally unilaterally approximated by $\operatorname{sing}_{\mathcal{T}}(\mathcal{S})$ by Lemma 3.2.3. We then apply Corollary 3.1.9 to get the dimension bound.

□

In the next section, we will give an application of Corollary 3.3.3 to the study of asymptotically optimally doubling measures (see the last claim of Theorem 4.3.4). We provide below an example motivated by the Plateau problem.

Example. Consider the class of 2 minimal cones in $\mathbb{R}^3$ as classified by Jean Taylor which were discussed in the introduction. Denote this class by $\mathcal{M}$. This class has three types of sets; planes, Y-type sets, and T-type (or tetrahedral) sets displayed in Figure 1.1, reproduced here as Figure 3.1. The class $\mathcal{M}$ contains all translates of the minimal cones passing

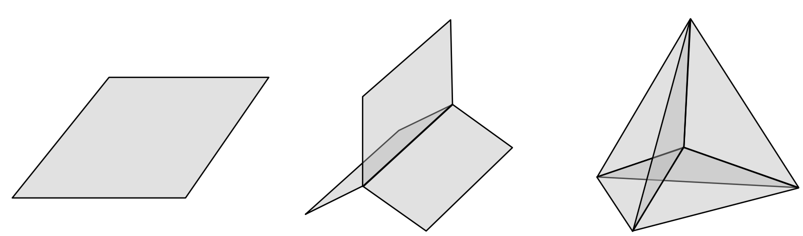

Figure 3.1: The 2 minimal cones in $\mathbb{R}^3$ (a reproduction of Figure 1.1)

through the origin. First, we note that $\mathcal{G}$, the collection of 2-planes, is detectable in $\mathcal{M}$. We note also that the nonflat points of Y-type sets are lines and the nonflat points of T-type sets are a union of 4 rays meeting at a single point. It follows that $\operatorname{sing}_{\mathcal{G}}(\mathcal{M})$ has a 1-Minkowski profile. Moreover, let $\mathcal{Y}$ be the subclass of all planes and Y-type sets. Then $\mathcal{Y}$ is detectable

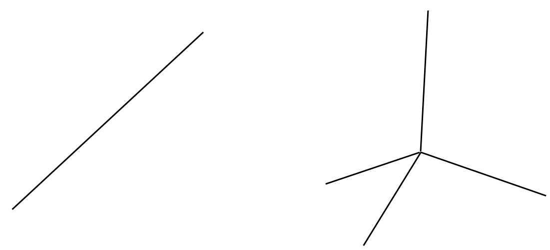

Figure 3.2: The class $\operatorname{sing}_{\mathcal{G}}(\mathcal{M})$.

in $\mathcal{M}$, and $\operatorname{sing}_{\mathcal{Y}}(\mathcal{M}) = \{\{0\}\}$ is the trivial local approximation class containing only the set which consists of the origin.

Thus, by Corollary 3.3.3, we have that if A is locally well approximated by $\mathcal{M}$, then $\overline{\dim}(A_{\mathcal{M}\setminus\mathcal{G}}) \leq 1$ and $\overline{\dim}(A_{\mathcal{M}\setminus\mathcal{Y}}) = 0$.

Chapter 4

LOCAL STRUCTURE OF ASYMPTOTICALLY OPTIMALLY DOUBLING MEASURES

4.1 Introduction

In this chapter, we study the relationship between the optimal doubling properties of a measure and the regularity and geometry of its support. This question had been considered in [9] and [29] where one of their crucial hypothesis was a baseline assumption of flatness. Roughly speaking they showed that if a Radon measure doubles asymptotically like Lesbegue measure of the appropriate dimension and the support of the measure is sufficiently flat then it can be locally parameterized as the image of an open set of the plane. Their study leaves open the question of what happens in the presence of non-flat points. In this chapter, we use the theory of Chapters 2 and 3 to give a local structure theorem for the support of an asymptotically optimally doubling measure.

An $(n-1)$-uniform measure on $\mathbb{R}^n$ is one for which the measure of any ball of radius r centered in the support is the same as $(n-1)$ Lebesgue measure a ball of radius r in $\mathbb{R}^{n-1}$, $\omega_{n-1}r^{n-1}$. Kowalski and Preiss showed in [21] that an $(n-1)$-uniform measure on $\mathbb{R}^n$ is, up to translation and rotation, surface measure on either a hyperplane or the cone $\mathrm{C} = \{x_1^2 + x_2^2 + x_3^2 = x_4^2\}$ (hereforward called a KP cone). An $(n-1)$-asymptotically optimally doubling measure is one whose asymptotic doubling properties coincide with $(n-1)$-dimensional Lebesgue measure (see Definition 4.1.3). Our first main result in Section 3 is to show that the support of an $(n-1)$-asymptotically optimally doubling measure at a nonflat point is well approximated by a (translated and rotated) KP cone (see Definition 4.1.1). In [20], the authors show that the (pseudo)tangent measures of an $(n-1)$-asymptotically optimally doubling measure are $(n-1)$-uniform. Coupling this with the classification of [21], the appearance of KP cones should not be surprising in the context of this chapter.

Definition 4.1.1. *Let $n \geq 4$. We define a* KP cone based at $y \in \mathbb{R}^n$ *to be a set* C *such*

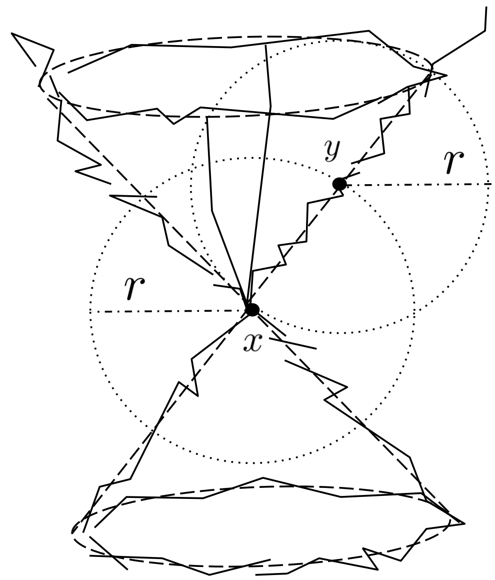

Figure 4.1: A set Σ such that $\Theta_\Sigma^{\mathcal{C}_0}(x,r)$ and $\Theta_\Sigma^{\mathcal{C}}(y,r)$ are small, $\Theta_\Sigma^{\mathcal{C}_0}(y,r)$ big.

that in some orthonormal coordinates $(x_1,\ldots,x_n)$ *centered at the origin,*

$$\mathrm{C} - y = \{x_4^2 = x_1^2 + x_2^2 + x_3^2\}.$$

We define the local approximation class $\mathcal{C}$ *to be the collection of all KP cones containing the origin. We define the local approximation class* $\mathcal{C}_0$ *to be the collection of all KP cones centered at the origin. Further, we let* $\mathcal{G} = \mathcal{G}(n, n-1)$ *be the local approximation class of* $(n-1)$*-spaces.*

We define the *support of* μ as

$$\mathrm{spt}(\mu) = \{x : \mu(B(x,r)) > 0 \text{ for all } r > 0\}. \tag{4.1}$$

The support may be alternatively viewed as the minimal closed set of comeasure 0. We will use the quantities $\Theta^{\mathcal{C}}$, $\Theta^{\mathcal{C}_0}$, and $\Theta^{\mathcal{G}}$ at different points of this analysis to obtain information about the local structure of $\mathrm{spt}(\mu)$.

Remark 4.1.2. *It is not hard to see that* $\overline{\mathcal{C}} = \mathcal{C} \cup \mathcal{G}$. *Indeed, suppose that* $\mathrm{C}_i \in \mathcal{C}$, *and* $\mathrm{C}_i \to S$ *for some set* S. *Let* x_i *be a nearest point to the origin such that* C_i *is based at* x_i. *It follows that either* x_i *are bounded, in which case* S *is a* KP *cone, or* $x_i \to \infty$, *in which case* S *is a plane.*

Because $\overline{\mathcal{C}} = \mathcal{C} \cup \mathcal{G} \supseteq \mathcal{G}$, *we have that for any set* A, *point* $x \in A$, *and radius* $r > 0$,

$$\Theta_A^{\mathcal{C}}(x,r) = \Theta_A^{\overline{\mathcal{C}}}(x,r) \leq \Theta_A^{\mathcal{G}}(x,r).$$

We now give increasingly strong conditions on the regularity of a measure μ.

Definition 4.1.3. *Let* μ *be a nonzero Radon measure on* $\mathbb{R}^n$ *and* $\Sigma = \operatorname{spt}\mu$.

(1) We say μ *is* locally doubling at $x \in \Sigma$ *if there exists a neighborhood* U *of* x *and a constant* C *such that for all* $y \in \Sigma \cap U$ *and all* $r > 0$ *such that* $B(y,r) \subseteq U$,

$$\frac{\mu(B(y,r))}{\mu(B(y,r/2))} \leq C.$$

(2) We say that μ *is* locally doubling *if it is locally doubling at all* $x \in \Sigma$.

(3) For $x \in \Sigma$, $r > 0$, *and and integer* $0 < m \leq n$ *(understood) we define*

$$R(\mu,x,r) = \sup\left\{\left|\frac{\mu(B(x,\tau r'))}{\mu(B(x,r'))} - \tau^m\right| : 0 < r' \leq r, \tau \in [1/2,1]\right\}. \tag{4.2}$$

For $K \subseteq \Sigma$, *we define*

$$R(\mu,K,r) = \sup_{x\in K} R(\mu,x,r). \tag{4.3}$$

(4) For an integer $0 < m \leq n$, *we say that* μ *is* m-asymptotically optimally doubling *if for all compact sets* $K \subseteq \Sigma$ *and* $\delta > 0$, *there exists a radius* $r_K > 0$ *such that*

$$R(\mu,K,r_K) < \delta. \tag{4.4}$$

That is, for all $x \in K$, $0 < r \leq r_K$ *and* $\tau \in [1/2,1]$,

$$\left|\frac{\mu(B(x,\tau r))}{\mu(B(x,r))} - \tau^m\right| < \delta. \tag{4.5}$$

Equivalently, we may say that the quantity $\frac{\mu(B(x,\tau r))}{\mu(B(x,r))} - \tau^m$ *converges to* 0 *uniformly on compact sets as* $r \downarrow 0$ *(independent of* $\tau \in [1/2,1]$*). In the case where* m *is understood, we will drop it from the beginning.*

*(5) For an integer $0 < m \leq n$, we say that μ is m-*uniform *if for all $x \in \Sigma$, $r > 0$,*

$$\mu(B(x,r)) = \omega_m r^m, \tag{4.6}$$

where $\omega_m = \mathcal{L}^m(B^m(0,1))$.

We now give precise statements of the theorems mentioned earlier. We begin with a theorem of David, Kenig, and Toro [9] which says that if Σ is assumed a priori to be $1/(2\sqrt{2})$-Reifenberg flat, then it is Reifenberg vanishing.

Theorem 4.1.4 ([9])**.** *Suppose that μ is an $(n-1)$-asymptotically optimally doubling measure on $\mathbb{R}^n$, and $\Sigma = \operatorname{spt}\mu$. If $n > 3$, suppose also that Σ is $1/(2\sqrt{2})$-Reifenberg flat. Then Σ is Reifenberg flat with vanishing constant.*

In [9], the authors showed a similar statement for arbitrary m (i.e., arbitrary codimension). However, since $m = n-1$ (codimension 1) will be our focus, we omit the generalization (which can be found in [9]). We expand their study from the specialized setting of flat points to all $(n-1)-$asymptotically optimally doubling measures and show a global statement akin to Reifenberg flatness, as well as showing that at a nonflat point x the support is well approximated by a KP cone based at x.

Theorem 4.1.5. *Suppose that μ is an $(n-1)$-asymptotically optimally doulbing measure on $\mathbb{R}^n$ and $\Sigma = \operatorname{spt}\mu$. Then*

$$\Theta_\Sigma^{\overline{\mathcal{C}}}(x,r) \to 0 \quad \text{as} \quad r \downarrow 0$$

uniformly on compact subsets (see Remark 4.1.2). That is, Σ is locally well approximated by $\overline{\mathcal{C}}$. Further, we have that for nonflat points x,

$$\Theta_\Sigma^{\mathcal{C}_0}(x,r) \to 0 \quad \text{as } r \downarrow 0$$

uniformly on compact sets.

In Section 2, we investigate the geometry of sets well approximated by $\overline{\mathcal{C}}$, using heavily the machinery from Chapter 2. In particular, we investigate the behavior of a set Σ under different assumptions on $\Theta_\Sigma^{\mathcal{C}_0}$, $\Theta_\Sigma^{\mathcal{C}}$, and $\Theta_\Sigma^{\mathcal{G}}$ at differening locations and scales, often exploiting their interplay. In Section 3, we investigate $(n-1)-$asymptotically optimally doubling measures, culminating in Theorem 4.1.5.

4.2 The Local Structure of $(n-1)$-Asymptotically Optimally Doubling Measures

In this section, we show that the support of an $(n-1)$-asymptotically optimally doubling measure is locally well approximated by $\mathcal{C}$. This is done through a consideration of pseudotangents. In particular, we use Theorem 4.2.2 which was shown by Kenig and Toro [20].

First, we define pseudotangent measures, the measure theoretic analogue of pseudotangent sets. Define the map $T_{x,r}(y) = (y-x)/r$. For a measure μ on $\mathbb{R}^n$, define $\mu_{x,r} = \frac{1}{\mu(B(x,r))} T_{x,r\#}\mu$ to be the (rescaled) push forward measure under the map $T_{x,r}$. That is,

$$\mu_{x,r}(A) = \frac{\mu\,(rA + x)}{\mu(B(x,r))}.$$

We write $\mu_i \rightharpoonup \mu$ for a sequence of measures μ_i converging weakly to μ in the sense of Radon measures.

Definition 4.2.1. *Let μ and ν be nonzero Radon measures on $\mathbb{R}^n$. We say that ν is a* pseudotangent measure *of μ at x if $x \in \operatorname{spt}\mu$ and there exist a sequence $x_i \in \operatorname{spt}\mu$ such that $x_i \to x$, a sequence of positive numbers $r_i \to 0$, and a sequence of positive numbers c_i such that $c_i T_{x_i,r_i\#}\mu \rightharpoonup \nu$. We say that ν is a* tangent measure *if it is a pseudotangent measure with $x_i = x$ for all i, and we denote the set of tangent measures to μ at x by $\operatorname{Tan}(\mu, x)$.*

Pseudotangent measures were introduced by Kenig and Toro [20] as a generalization of tangent measures. The following theorem says roughly that the pseudotangents of a locally doubling measure behave as we would expect. The first part gives a normalization on c_i and the second part says that blow ups of the support converge to the support of the pseudotangent measure.

Theorem 4.2.2 ([20])**.** *Let μ be an asymptotically optimally doubling measure on $\mathbb{R}^n$, $\Sigma = \operatorname{spt}\mu$, and ν be a pseudotangent measure of μ with $c_i T_{x_i,r_i\#}\mu \rightharpoonup \nu$. Let $\Sigma_i = (\Sigma - x_i)/r_i = \operatorname{spt}\mu_{x_i,r_i}$. Then $\Sigma_i \to \Sigma_\infty = \operatorname{spt}\nu$ as $i \to \infty$.*

When μ is an m-asymptotically optimally doubling measure, the pseudotangent measures are (up to multiplication by a constant) m-uniform measures.

Theorem 4.2.3 ([20])**.** *Suppose that μ is an m-asymptotically optimally doubling measure on $\mathbb{R}^n$, and ν is a pseudotangent measure of μ. Then up to multiplication by a constant, ν is an m-uniform measure. If $m = n-1$, the classification of [21] says that ν is $(n-1)$-dimensional Hausdorff measure restricted to either an $(n-1)$-plane containing 0 or a KP cone containing 0.*

Remark 4.2.4. *In light of Theorem 4.2.3, it is helpful to recall that KP cones are defined in $\mathbb{R}^n$ only for $n \geq 4$.*

Corollary 4.2.5. *Let μ be an $(n-1)$-asymptotically optimally doubling measure on $\mathbb{R}^n$ and $\Sigma = \mathrm{spt}(\mu)$. Then for all $x \in \Sigma$, $\Psi\text{-}\mathrm{Tan}(\Sigma, x) \subseteq \overline{\mathcal{C}}$. Equivalently, Σ is locally well approximated by $\overline{\mathcal{C}}$.*

4.3 The Geometry of Sets Locally Well Approximated by $\overline{\mathcal{C}}$

The goal of this section is to analyze the geometry of sets which are well approximated by KP cones. For a set Σ with a tangent plane at a, let $T_a\Sigma$ be the tangent plane to Σ at a. We use the convention that the tangent plane $T_a\Sigma$ is centered at a; that is, $a \in T_a\Sigma$.

Lemma 4.3.1. *Let C be a KP cone in $\mathbb{R}^4$ based at 0. There exists C_0 such that for any $a \in \mathrm{C}$, we have that exactly one of the following holds:*

(1) $a = 0$, in which case $\Theta^{\mathcal{G}}_{\mathrm{C}}(a, r) = 1/\sqrt{2}$ for all $r > 0$,

(2) or $a \neq 0$, in which case for all $0 < r \leq |a|/2$, $\mathrm{D}^{a,r}[\mathrm{C}, T_a\mathrm{C}] \leq C_0 r/|a|$ (and thus $\Theta^{\mathcal{G}}_{\mathrm{C}}(a, r) \leq C_0 r/|a|$).

Proof. Claim (1) follows from an elementary computation. Let $a \in \mathrm{C}, |a| = 1$. Note that $\mathrm{C} \setminus B(0, 1/2)$ is a smooth manifold, and hence there exists a constant C_0 such that for all $0 < r \leq 1/2$, $\mathrm{D}^{a,r}[\mathrm{C}, T_a\mathrm{C}] \leq C_0 r$. For any other point $b \in \mathrm{C}$, $|b| = 1$, there is an isometry fixing 0, taking a to b, and taking C to C. Hence, for all $|a| = 1, a \in \mathrm{C}$, $0 < r \leq 1/2$, $\mathrm{D}^{a,r}[\mathrm{C}, T_a\mathrm{C}] \leq C_0 r$. For $a \neq 0$, set $b = a/|a| \in \mathrm{C}$. Then $\mathrm{D}^{b,r}[\mathrm{C}, T_b\mathrm{C}] \leq C_0 r$ for $0 < r \leq 1/2$. Because C is a cone based at 0 and $|a|b = a$, we get that $|a|\mathrm{C} = \mathrm{C}$ and $|a|T_b\mathrm{C} = T_a\mathrm{C}$. This gives that $\mathrm{D}^{a,|a|r}\left[\mathrm{C}, T_a\mathrm{C}\right] = \mathrm{D}^{b,r}\left[\mathrm{C}, T_b\mathrm{C}\right] \leq C_0 r$ for all $0 < r \leq 1/2$. Plugging in $r/|a|$ for r, we get that

$$\mathrm{D}^{a,r}\left[\Sigma, \mathrm{C}\right] \leq C_0 \frac{r}{|a|}$$

for all $0 < r \leq |a|/2$, and hence have proven (2). □

Remark 4.3.2. *Using Lemma 4.1 of [1], one can obtain that in Lemma 4.3.1, $C_0 = 1$.*

Corollary 4.3.3. *By Lemma 4.3.1, $\mathcal{G}$ is $(1/2\sqrt{2}, C_0 s)$ detectable in $\overline{\mathcal{C}}$.*

From detectability, we get the following structure theorem for the support of asymptotically optimally doubling measures. Note that these results improve on the author's original partial result, [23] Theorem 3.8. The improvement is due to replacing a connectedness at infinity argument (which yields pointwise information) by a detectability argument (which yields locally uniform results).

Theorem 4.3.4. *Suppose that μ is an $(n-1)-$asymptotically optimally doubling measure on $\mathbb{R}^n$ with $\Sigma = \operatorname{spt}\mu$. Then Σ is locally well approximated by $\overline{\mathcal{C}}$. Moreover, we can write Σ as a disjoint union*

$$\Sigma = \Sigma_{\mathcal{G}} \cup \Sigma_{\mathcal{C}_0}, \quad (\Sigma_{\mathcal{G}} \cap \Sigma_{\mathcal{C}_0} = \emptyset)$$

such that:

- *$\Sigma_{\mathcal{G}}$ is relatively open in Σ and locally well approximated by $\mathcal{G}$.*
- *$\Sigma_{\mathcal{C}_0}$ is closed and Σ is locally well approximated by $\mathcal{C}_0$ along $\Sigma_{\mathcal{C}_0}$.*
- *$\overline{\dim}(\Sigma_{\mathcal{C}_0}) \leq n-4$.*

Proof. The fact that Σ is locally well approximated by $\overline{\mathcal{C}}$ is established by Corollary 4.2.5. By Corollary 4.3.3, $\mathcal{G}$ is detectable in $\overline{\mathcal{C}}$. The first two bullets follow from Theorem 2.5.2 (Open/Closed Decompositions) and Corollary 2.5.4. In particular, $\Sigma_{\mathcal{C}\setminus\mathcal{G}} = \Sigma_{\mathcal{C}_0}$. Note that $\operatorname{sing}_{\mathcal{G}}(\overline{\mathcal{C}}) = \mathcal{G}(n, n-4)$, the co-dimension 4 Grassmannian. Thus, $\operatorname{sing}_{\mathcal{G}}(\overline{\mathcal{C}})$ has an $(n-4)$-Minkowski profile. By Corollary 3.3.3, we get that $\overline{\dim}(\Sigma_{\mathcal{C}_0}) \leq n-4$. □

Chapter 5

HIGHER REGULARITY CASE STUDY: NONFLAT POINTS OF HÖLDER ASYMPTOTICALLY OPTIMALLY DOUBLING MEASURES

5.1 Introduction

In this chapter, we investigate via example the notion of higher regularity local set approximation. In the previous chapter, our optimal regularity was to show that the quantities $\Theta^{\mathcal{C}}$, $\Theta^{\mathcal{C}_0}$, and $\Theta^{\mathcal{G}}$ went to zero locally uniformly in the appropriate senses. We will now assume a stronger doubling condition on the measure, specifically that the error term $R(\mu, x, r)$ is bounded by Cr^{α} for r small enough (see Definition 4.1.3). David, Kenig, and Toro [9] studied this case under an additional assumption of flatness. By assuming additionally that the support was locally $1/(4\sqrt{2})$-approximable by $\mathcal{G}$ (or $1/(4\sqrt{2})$ Reifenberg flat), they show that the support is in fact a $C^{1,\beta}$ manifold; that is, in a neighborhood of every point, the support admits a $C^{1,\beta}$ parametrization by a plane. However, their study leaves open what happens in a neighborhood of a nonflat point.

We here complete the investigation in the case $n = 4$. We do this by showing an analogue of the result of David, Kenig and Toro for the nonflat points. Specifically, we show that every nonflat point of the support has a neighborhood in which the support admits a $C^{1,\beta}$ parametrization by a KP cone.

This is accomplished as follows. The stronger doubling condition will lead us to an approximation scheme with $\Theta^{\mathcal{C}}$, $\Theta^{\mathcal{C}_0}$, and $\Theta^{\mathcal{G}}$ bounded by Hölder error terms at appropriate scales. We will then use this information to explicitly construct our parametrization in a neighborhood, and perform a quantitative analysis on it similar to the analysis of Jean Taylor in [34], and prove that our map extends to be $C^{1,\beta}$ on an open neighborhood. Part of the goal of this analysis is to lay bare the techniques needed to construct a $C^{1,\beta}$ parametrization by a singular set.

A major difficulty in obtaining Hölder bounds on $\Theta^{\mathcal{C}}$, $\Theta^{\mathcal{C}_0}$, and $\Theta^{\mathcal{G}}$ lies in using the densities to capture the local structure of the measure. This is accomplished through using the moments of the measure which we introduce in the next section. Roughly speaking, the first moment captures how close a set is to a cone in some ball, and the second moment will capture information about the curvature. Although this analysis does not fit into the general scheme of local set approximation, it will hopefully serve as an interesting example of obtaining higher regularity bounds on local approximability.

We now give precise definitions. Recall from Definition 4.1.3 that for $x \in \Sigma$, $r > 0$, and and integer $0 < m \leq n$ (understood) we let

$$R(\mu, x, r) = \sup \left\{ \left| \frac{\mu(B(x, \tau r'))}{\mu(B(x, r'))} - \tau^m \right| : 0 < r' \leq r, \tau \in [1/2, 1] \right\}. \tag{5.1}$$

For $K \subseteq \Sigma$, we set

$$R(\mu, K, r) = \sup_{x \in K} R(\mu, x, r). \tag{5.2}$$

Definition 5.1.1. *Let μ be a Radon measure on $\mathbb{R}^n$.*

i. For $\alpha > 0$ and an integer $0 < m \leq n$, we say that μ is (α, m)-Hölder asymptotically optimally doubling *if for all compact sets $K \subseteq \Sigma$, there exist a constant C_K and a radius $r_0 > 0$ such that for $0 < r \leq r_0$,*

$$R(\mu, K, r) \leq C_K r^\alpha. \tag{5.3}$$

That is, for all $x \in K$, $0 < r \leq r_0$, $\tau \in [1/2, 1]$,

$$\left| \frac{\mu(B(x, \tau r))}{\mu(B(x, r))} - \tau^m \right| \leq C_K r^\alpha. \tag{5.4}$$

In the case where α and m are understood, we will drop them from the beginning.

ii. For $\alpha > 0$ and an integer $0 < m \leq n$, we say μ is (α, m)-Hölder asymptotically uniform *if for each compact set $K \subseteq \Sigma$, there exist a constant C_K and a radius $r_0 > 0$ such that for all $x \in K$ and $0 < r \leq r_0$,*

$$\left| \frac{\mu(B(x, r))}{\omega_m r^m} - 1 \right| \leq C_K r^\alpha, \tag{5.5}$$

where $\omega_m = \mathcal{L}^m(B^m(0, 1))$.

Although Definition 5.1.1(1) gives a stronger property than Definition 5.1.1(2), we observe the following partial converse. We use the notation $\mu \llcorner g$ to be the measure $\mu \llcorner g(A) = \int_A g \, d\mu$.

Lemma 5.1.2 ([9],[29]). *Let μ be an $(\alpha, m)-$Hölder asymptotically optimally doubling measure on $\mathbb{R}^n$. Then the density $\theta(x) = \lim_{r \downarrow 0} \theta(x, r)$ exists and is finite and nonzero at μ-almost every $x \in \mathbb{R}^n$ and $\nu = \mu \llcorner \frac{1}{\theta}$ is $(\frac{\alpha}{\alpha+1}, m)$-Hölder asymptotically uniform.*

As a consequence of Lemma 5.1.2, we note that to study the support of Hölder asymptotically optimally doubling measures, we can study the support of Hölder asymptotically uniform measures. We note, however, that this is not true in general for asymptotically optimally doubling measures. That is, there are asymptotically optimally doubling measures whose measure is not given by the density of the set. This follows from a theorem of David, Kenig, and Toro [9] which states that a closed set $\Sigma \subseteq \mathbb{R}^n$ is the support of an $(n-1)$ asymptotically optimally doubling measure μ if and only if Σ is Reifenberg vanishing (that is, locally well approximated by $\mathcal{G}$).

Remark 5.1.3. *In the coming sections, we will have occasion to use the quantities $\mathrm{d}^{\mathrm{x,r}}$, $\widetilde{\mathrm{d}}^{x,r}$, $\mathrm{D}^{\mathrm{x,r}}$, and $\widetilde{\mathrm{D}}^{x,r}$. The extra technical complications which come from using relative Walkup-Wets distance in conjunction with relative Hausdorff distance are outlined in Appendix A. However, the key point is that if C is a cone with vertex at x, then all of these quantities are equivalent up to a factor of two (Lemma A.1.2) and relative Hausdorff distance satisfies a near monotonicity formula (Lemma A.1.3).*

5.2 Control of Density and Moments About a Nonflat Point

In this section, we begin our investigation of the nonflat points in the support of a Hölder asymptotically uniform measure (see Theorem 5.1.2). We find appropriate Hölder estimates on $\Theta_\Sigma^{\mathcal{G}}$, $\Theta_\Sigma^{\mathcal{C}_0}$, and $\Theta_\Sigma^{\mathcal{C}}$ in a neighborhood of a nonflat point in the support. In the next section, we use these estimates to construct a parametrization by a KP cone. Stated precisely, in the next two sections we prove the following theorem.

Theorem 5.2.1. *For any $\alpha > 0$, there exists $\beta = \beta(\alpha)$ with the following property. If μ is an $(\alpha, 3)$-asymptotically uniform measure, then for any $x \in \operatorname{spt} \mu$ which is nonflat there* *exist a KP cone centered at* 0*, neighborhoods* U *of* 0 *and* U' *of* x *and a diffeomorphism* $\varphi \in C^{1,\beta}(U \to U')$ *such that* $\varphi(\mathrm{C} \cap U) = \mathrm{spt}(\mu) \cap U'$*. Further,* φ *has the property that* $\varphi(0) = x$ *and* $D_0\varphi = \mathrm{Id}$.

To this end, we assume that $\alpha > 0$ and

$$\begin{aligned} &\mu \text{ is a Radon measure on } \mathbb{R}^4 \text{ which is } (\alpha,3)\text{−asymptotically uniform,} \\ &\Sigma = \operatorname{spt}\mu \text{ satisfies that } 0 \in \Sigma \text{ is a nonflat point,} \\ &\left|\tfrac{\mu(B(x,r))}{\omega_3 r^3} - 1\right| \le C_0 r^\alpha \text{ for } x \in \Sigma \cap B(0,1), 0 < r \le 1 \text{ (see Definition 4.1.3).} \end{aligned} \tag{5.6}$$

The second and third conditions may be viewed simply as a translation and dilation to normalize the scales at which we work. In Section 5.2.1, we adopt methods from [9] to describe the *moments of* μ, which are polynomials related to the structure of μ. In Section 5.2.2, we use the information about the moments to get quantitative bounds on $\Theta_\Sigma^{\mathcal{C}_0}(0,r)$. Finally in Section 5.2.3, we develop quantitative information at points near the origin and scales sufficiently small. We then construct a parametrization in Section 5.3 for a set Σ satisfying the estimates we demonstrate. In this chapter, the constant C depends on α and C_0, and the radius r_0 is chosen small enough depending on α and C_0. We also choose r_0 small enough such that

$$\sup_{0<r\le r_0} \Theta^{\mathcal{C}_0}(0,r) \quad \text{and} \quad \sup_{\substack{0<r\le r_0 \\ x\in\Sigma\cap B(0,r_0)}} \Theta^{\mathcal{C}}(x,r)$$

can be chose small enough (see Theorem 4.3.4).

5.2.1 Control of the Moments

Let ν be a Radon measure on $\mathbb{R}^n$. Define the *first moment* of ν at a point $x \in \mathrm{spt}(\nu)$ and a scale r to be the vector

$$b_{x,r}(\nu) = \frac{n+1}{2\omega_{n-1} r^{n+1}} \int_{B(x,r)} (r^2 - |y-x|^2)(y-x)\, d\mu(y). \tag{5.7}$$

Define the *second moment* of ν at a point $x \in \mathrm{spt}(\nu)$ and a scale r to be the quadratic form

$$Q_{x,r}(\nu)(y) = \frac{n+1}{\omega_{n-1} r^{n+1}} \int_{B(x,r)} \langle y, z-x\rangle^2\, d\mu(z). \tag{5.8}$$

Define also the *trace of* $Q_{x,r}(\nu)$ to be

$$\mathrm{tr}Q_{x,r}(\nu) = \frac{n+1}{\omega_{n-1}r^{n+1}} \int_{B(x,r)} |z-x|^2 \, d\mu(z). \tag{5.9}$$

In any orthonormal coordinates centered at the origin $(x_1, \ldots, x_n)$ (and x and r understood), we set

$$q_{ij}(\nu) = \frac{n+1}{\omega_{n-1}r^{n+1}} \int_{B(x,r)} (z_i - x_i)(z_j - x_j) \, d\nu(z). \tag{5.10}$$

It follows that $Q_{x,r}(\nu) = \sum_{i,j=1}^{n} q_{ij}(\nu) x_i x_j$. Moreover, $\mathrm{tr}Q_{x,r}(\nu) = \sum_{i=1}^{n} q_{ii}(\nu)$, and so coincides with the usual notion of the trace of a quadratic polynomial. For the rest of Section 4, we set $Q_{x,r} = Q_{x,r}(\mu)$ and $b_{x,r} = b_{x,r}(\mu)$ (see (5.6)).

First, we set some notation. Fix $0 < \gamma < \theta < \alpha/2$ for the remainder of Section 4. We will sometimes work at the scale $\rho = r^{1+\gamma}$. We also denote $\widetilde{Q}_{x,r}(z) = Q_{x,r}(z) - |z|^2$. Further, we set the blow up of Σ at scale r to be $\Sigma_r = (1/r)\Sigma$ (and so $\Sigma_\rho = (1/\rho)\Sigma$). We now summarize some of the results of [9] which highlight the interactions between the moment at the scale r and the geometry at a different scale.

Theorem 5.2.2. *Recall hypotheses (5.6).*

(1) [9] *For* $0 < r \leq 1/2$, $|\mathrm{tr}Q_{0,r} - (n-1)| \leq Cr^\alpha$.

(2) [9] *For* $0 < r \leq 1/2$. $|b_{0,r}| \leq Cr^{1+\theta}$.

(3) [9] *For* $0 < r \leq 1/2$ *and* $z \in \Sigma \cap B(0, r/2)$,

$$|2\langle b_{0,r}, z\rangle + \widetilde{Q}_{0,r}(z)| \leq C\frac{|z|^3}{r} + Cr^{2+\alpha} \tag{5.11}$$

(4) *For* $0 < r \leq 1/2$, $0 < s \leq r/4$, *and* $z \in \Sigma_s \cap B(0,2)$,

$$|\widetilde{Q}_{0,r}(z)| \leq C\frac{s}{r} + C\frac{r^{2+\alpha}}{s^2} + C\frac{r^{1+\theta}}{s}. \tag{5.12}$$

(5) *For* $M \geq 4$, $0 < r \leq 1/2$, *and* $z \in \Sigma_{r/M} \cap B(0,2)$

$$|\widetilde{Q}_{0,r}(z)| \leq CMr^\theta + \frac{C}{M} + CM^2 r^\alpha. \tag{5.13}$$

(6) For $\tau \in [1/2, 1]$, r such that $0 < r^\gamma \leq 1/8$, and $z \in \Sigma_\rho \cap B(0,2)$,

$$\left|\widetilde{Q}_{0,\tau r}(z)\right| \leq Cr^{\theta-\gamma} + Cr^{\gamma} + Cr^{\alpha-2\gamma}. \tag{5.14}$$

Letting $\beta_0 = \min(\theta-\gamma, \gamma, \alpha - 2\gamma)$, (5.14) gives that

$$\left|\widetilde{Q}_{0,\tau r}(z)\right| \leq Cr^{\beta_0}. \tag{5.15}$$

Remark 5.2.3. *We note that in [9], two cases with respect to (2) were considered: the case where $b_{0,r}$ is small (satisfies Theorem 5.2.2(2)), and the case where it is large (does not satisfy Theorem 5.2.2(2)). However, contained in their analysis, they showed that the latter case automatically implies flatness. Thus, our nonflatness assumption implies Theorem 5.2.2(2).*

Proof of (4), (5), and (6). We begin by proving (4). Set $x = sz$. Then $x \in \Sigma \cap B(0, 2s) \subseteq \Sigma \cap B(0, r/2)$. We apply (3) to see that

$$\left|2\left\langle \frac{b_{0,r}}{s}, z\right\rangle + \widetilde{Q}_{0,r}(z)\right| = \frac{1}{s^2}\left|2\langle b_{0,r}, x\rangle + \widetilde{Q}_{0,r}(x)\right| \leq \frac{1}{s^2}\left(C\frac{|x|^3}{r} + Cr^{2+\alpha}\right) \leq C\frac{s}{r} + c\frac{r^{2+\alpha}}{s^2}. \tag{5.16}$$

By applying (2), we get

$$|\widetilde{Q}_{0,r}(z)| \leq 2\frac{|b_{0,r}||z|}{s} + C\frac{s}{r} + C\frac{r^{2+\alpha}}{s^2} \leq C\frac{r^{1+\theta}}{s} + C\frac{s}{r} + C\frac{r^{2+\alpha}}{s^2} \tag{5.17}$$

We now have that (5) follows from (4) by setting $s = r/M$ and checking that $0 < s \leq r/4$. Similarly, (6) follows from (4) by taking $s = \rho = r^{1+\gamma}$ and taking τr as our radius, and checking that $0 < s \leq \tau r/4$.

□

We now seek to understand the second moment $Q_{0,r}(x)$. Let (x_1, x_2, x_3, x_4) be orthonormal coordinates centered at the origin. Since $Q_{0,r}$ is a quadratic polynomial, we can represent it as the matrix $Q_{0,r} = (q_{ij})$. Note that if we compute the second moment $Q_{0,r}(\mathcal{H}^3 \llcorner \mathrm{C}) = K$ of 3-dimensional Hausdorff measure on the KP cone $\mathrm{C} = \{x_4^2 =$

$x_1^2 + x_2^2 + x_3^2\}$ (at any radius) we get

$$K = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

Our first lemma proves that at small enough radii, $Q_{0,r}$ becomes close to K.

Lemma 5.2.4. *Let $\delta > 0$. There exists an $r_0 > 0$ such that for all $0 < r \leq r_0$, there exists an orthonormal basis (x_1, x_2, x_3, x_4) for which $\max_{ij} |q_{ij} - K_{ij}| < \delta$.*

Proof. Let $\delta > 0$ be given. Fix parameters $M \geq 4$ and $\varepsilon > 0$ to be specified later. By Theorem 4.3.4, there exists $r_0 > 0$ such that for all $0 < r \leq r_0$,

$$\Theta_{\Sigma}^{\mathcal{C}_0}(0, r) < \frac{\varepsilon}{M}. \tag{5.18}$$

Fix $0 < r \leq r_0$. Let C be a KP cone centered at the origin such that

$$\mathrm{D}^{0,r}[\Sigma, \mathrm{C}] < \frac{\varepsilon}{M}. \tag{5.19}$$

Let (x_1, x_2, x_3, x_4) be orthonormal coordinates such that

$$\mathrm{C} = \{x_4^2 = x_1^2 + x_2^2 + x_3^2\}. \tag{5.20}$$

By finding the vector from x to C normal to C, a quick computation shows that for any point $x \in \mathbb{R}^4$,

$$\mathrm{dist}(x, \mathrm{C}) = \frac{\big||x_4| - |(x_1, x_2, x_3)|\big|}{\sqrt{2}}. \tag{5.21}$$

We manipulate (5.21) to get

$$2\,\mathrm{dist}(x, \mathrm{C})^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2|x_4||(x_1, x_2, x_3)|. \tag{5.22}$$

Set $C_1 = 5/\omega_3$, we compute

$$\begin{aligned} \frac{C_1}{r^5} \int_{B(0,r)} 2\,\mathrm{dist}(x, \mathrm{C})^2 \, d\mu(x) &= \frac{C_1}{r^5} \int_{B(0,r)} x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2|x_4||(x_1, x_2, x_3)| \, d\mu(x) \\ &= \mathrm{tr} Q_{0,r} - 2\frac{C_1}{r^5} \int_{B(0,r)} |x_4||(x_1, x_2, x_3)| \, d\mu(x). \end{aligned} \tag{5.23}$$

We apply Hölder's inequality to find

$$\begin{aligned}\frac{C_1}{r^5}\int_{B(0,r)}|x_4||(x_1,x_2,x_3)|\,d\mu &\le \left(\frac{C_1}{r^5}\int_{B(0,r)}|x_4|^2\,d\mu\right)^{\frac{1}{2}}\left(\frac{C_1}{r^5}\int_{B(0,r)}|(x_1,x_2,x_3)|^2\,d\mu\right)^{\frac{1}{2}}\\ &= \sqrt{(q_{44})(q_{11}+q_{22}+q_{33})}\end{aligned} \tag{5.24}$$

Substituting (5.24) into (5.23) and manipulating, we get that

$$\frac{C_1}{r^5}\int_{B(0,r)}2\,\mathrm{dist}(x,\mathrm{C})^2\,d\mu \ge \mathrm{tr}Q_{0,r}-2\sqrt{(q_{11}+q_{22}+q_{33})(q_{44})} = \left(\sqrt{q}_{44}-\sqrt{q_{11}+q_{22}+q_{33}}\right)^2. \tag{5.25}$$

By applying the Hölder asymptotically uniform property and (5.19), we also have that

$$\begin{aligned}\frac{C_1}{r^5}\int_{B(0,r)}2\,\mathrm{dist}(x,\mathrm{C})^2\,d\mu &\le \frac{2C_1}{r^5}\int_{B(0,r)}\left(\frac{\varepsilon r}{M}\right)^2 d\mu = 10\frac{\mu(B(0,r))}{\omega_3 r^3}\frac{\varepsilon^2}{M^2}\\ &\le 10\left(1+C_0 r_0^{\alpha}\right)\frac{\varepsilon^2}{M^2} \le C\frac{\varepsilon^2}{M^2}.\end{aligned} \tag{5.26}$$

Combining (5.25) and (5.26), we have that

$$\left(\sqrt{q}_{44}-\sqrt{q_{11}+q_{22}+q_{33}}\right)^2 \le C\frac{\varepsilon^2}{M^2}. \tag{5.27}$$

Note that $0 \le q_{ii} \le 3 + Cr^{\alpha}$ by Theorem 5.2.2(1) and the definition of q_{ij}. We use this fact and (5.27) to get

$$\left|q_{44}-q_{11}-q_{22}-q_{33}\right| = \left|\sqrt{q_{44}}-\sqrt{q_{11}+q_{22}+q_{33}}\right|\cdot\left|\sqrt{q_{44}}+\sqrt{q_{11}+q_{22}+q_{33}}\right| \le C\frac{\varepsilon}{M} \tag{5.28}$$

By Theorem 5.2.2(1), we also have that

$$|q_{11}+q_{22}+q_{33}+q_{44}-3| \le Cr^{\alpha}. \tag{5.29}$$

From (5.28) and (5.29), we get that

$$|q_{44}-\frac{3}{2}| \le C\frac{\varepsilon}{M}+Cr^{\alpha} \quad \text{and} \quad |q_{11}+q_{22}+q_{33}-\frac{3}{2}| \le C\frac{\varepsilon}{M}+Cr^{\alpha}. \tag{5.30}$$

Set $\sigma = \sigma(r, M, \varepsilon) := C\varepsilon/M + Cr_0^{\alpha}$ for the constants above. Note $\sigma \to 0$ as $r, \varepsilon \to 0$ (recall $M \ge 4$).

From Theorem 5.2.2, we have that

$$|\widetilde{Q}_{0,r}(z)| \le CMr^{\theta} + \frac{C}{M} + CM^2r^{\alpha} \quad \text{for } z \in \Sigma_{r/M}\cap B(0,2). \tag{5.31}$$

We now extend this to information about the points in C. First, we note that because C is a cone centered at the origin, by Lemma A.1.2 and (5.19), it follows that

$$\mathrm{D}^{0,1}\left[\mathrm{C}, \Sigma_{r/M}\right] \leq 2\varepsilon. \tag{5.32}$$

Since $\widetilde{Q}_{0,r}$ is a quadratic form with $|\widetilde{Q}_{0,r}(x)| \leq C|x|^2$, we get that for any $e \in \mathbb{S}^3$,

$$|\partial_e \widetilde{Q}_{0,r}(x)| \leq C|x| \leq C \quad \text{ for } x \in B(0,2). \tag{5.33}$$

Let $a \in \mathrm{C} \cap B(0,1)$. By (5.32) there is a point $x \in \Sigma_{r/M}$ such that $|x-a| < 2\varepsilon$. Setting $e = \frac{x-a}{|x-a|}$, we integrate along the path from x to a along e, apply (5.33), and get that

$$|\widetilde{Q}_{0,r}(a)| \leq |\widetilde{Q}_{0,r}(x)| + |\widetilde{Q}_{0,r}(a) - \widetilde{Q}_{0,r}(x)| \leq |\widetilde{Q}_{0,r}(x)| + C\varepsilon. \tag{5.34}$$

Combining (5.31) and (5.34), we get that

$$|\widetilde{Q}_{0,r}(a)| \leq CMr^\theta + \frac{C}{M} + CM^2 r^\alpha + C\varepsilon =: \eta(r, M, \varepsilon) \quad \text{ for } a \in \mathrm{C} \cap B(0,1). \tag{5.35}$$

Note that by choosing ε small, M large, and r very small (depending on M), we can make $\eta = \eta(r, M, \varepsilon)$ as small as we like.

We now plug in some special points of C to extract information about $Q_{0,r}$. We continue to work in the same orthonormal coordinates (x_1, x_2, x_3, x_4) (see (5.20)). Let $z_i^{\pm} = (x_i \pm x_4)/\sqrt{2}$ for $i = 1, 2, 3$. Because $z_i^{\pm} \in \mathrm{C} \cap B(0,1)$, we may apply (5.35) to get

$$|\widetilde{Q}_{0,r}(z_i^{\pm})| = \left|\frac{1}{2}q_{ii} + \frac{1}{2}q_{44} - 1 \pm q_{i4}\right| \leq \eta \tag{5.36}$$

(recall that by (5.10), $q_{i4} = q_{4i}$). From (5.36) and (5.30), we get

$$\left|q_{ii} - \frac{1}{2}\right| \leq \left|\frac{1}{2}q_{ii} + \frac{1}{2}q_{44} - 1 + q_{i4}\right| + \left|\frac{1}{2}q_{ii} + \frac{1}{2}q_{44} - 1 - q_{i4}\right| + \left|\frac{3}{2} - q_{44}\right| \leq 2\eta + \sigma. \tag{5.37}$$

We also get from (5.36) that

$$|q_{i4}| \leq \frac{1}{2}\left|q_{i4} + \frac{1}{2}q_{ii} + \frac{1}{2}q_{44} - 1\right| + \frac{1}{2}\left|q_{i4} - \frac{1}{2}q_{ii} - \frac{1}{2}q_{44} + 1\right| \leq \eta. \tag{5.38}$$

Let $y_{ij} = (x_i + x_j)/2 + x_4/\sqrt{2}$ for $i, j = 1, 2, 3$. Because $y_{ij} \in \mathrm{C} \cap B(0,1)$, we may apply (5.35) to get

$$|\widetilde{Q}_{0,r}(y_{ij})| = \left|\frac{1}{4}q_{ii} + \frac{1}{4}q_{jj} + \frac{1}{2}q_{44} + \frac{1}{2}q_{ij} + \frac{1}{\sqrt{2}}q_{i4} + \frac{1}{\sqrt{2}}q_{j4} - 1\right| \leq \eta \tag{5.39}$$

(recall that by (5.10), $q_{ij} = q_{ji}$). So from (5.30), (5.37), (5.38), and (5.39), we get

$$|q_{ij}| \leq 2\eta + \left|q_{ii} - \frac{1}{2}\right| + \left|q_{jj} - \frac{1}{2}\right| + 2\left|q_{44} - \frac{3}{2}\right| + 2\sqrt{2}|q_{i4}| + 2\sqrt{2}|q_{j4}| \leq C\eta + C\sigma. \quad (5.40)$$

Hence, by (5.30), (5.37), (5.38), and (5.40), we can make η and σ small enough such that $\max_{ij} |q_{ij} - K_{ij}| \leq \delta$ in the orthonormal basis (x_1, x_2, x_3, x_4).

□

Lemma 5.2.5. *For any $\delta > 0$, there is an r_0 small enough such that the following hold. For $0 < r \leq r_0$, there is an orthonormal basis (x_1, x_2, x_3, x_4) diagonalizing $Q_{0,r}$ as*

$$Q_{0,r} = \sum_{i=1}^{4} \lambda_i x_i^2 \quad (5.41)$$

and

$$|\lambda_i - \frac{1}{2}| < \delta \text{ for } i = 1, 2, 3 \quad \text{and} \quad |\lambda_4 - \frac{3}{2}| < \delta. \quad (5.42)$$

Proof. Let $\delta > 0$ be given. Let $\eta > 0$ to be chosen small enough. Let r_0 be small enough that Lemma 5.2.4 is satisfied with η in place of δ. Fix $0 < r \leq r_0$, and let (y_1, y_2, y_3, y_4) be the orthonormal basis given by Lemma 5.2.4, i.e., such that

$$\max_{ij} |q_{ij} - K_{ij}| \leq \eta. \quad (5.43)$$

Note that the eigenvalues of K are $1/2$ with multiplicity 3 and $3/2$ with multiplicity 1. Note that $Q_{0,r}$ has real eigenvalues because it is symmetric. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ be the eigenvalues of $Q_{0,r}$. By the theory of Gershgorin disks [see, for example [4]], there exists η small enough such that (5.42) is satisfied. Further, because $Q_{0,r}$ is symmetric, there is an orthonormal basis (x_1, x_2, x_3, x_4) diagonalizing $Q_{0,r}$ such that (5.41) is satisfied.

□

We now know the structure of $Q_{0,r}$ for r small enough. Recall that $\widetilde{Q}_{0,r} = Q_{0,r} - |\cdot|^2$. We now give a lemma which will connect the unilateral approximability of a set Σ with its values on the polynomials $\widetilde{Q}_{0,r}$.

Lemma 5.2.6. *Let (x_1, x_2, x_3, x_4) be orthonormal coordinates on $\mathbb{R}^4$ and $P(x) : \mathbb{R}^4 \to \mathbb{R}$ be a polynomial of the form*

$$P(x) = \sum_{i=1}^{4} \eta_i x_i^2 \tag{5.44}$$

such that

$$\begin{aligned} |\eta_i + \tfrac{1}{2}| < \tfrac{1}{8} \quad \textit{for } i \in \{1,2,3\} \\ |\eta_i - \tfrac{1}{2}| < \tfrac{1}{8} \quad \textit{for } i = 4 \end{aligned} \tag{5.45}$$

Let Σ be a set such that

$$|P(x)| \leq \varepsilon \textit{ for all } x \in \Sigma \cap B(0,1), \tag{5.46}$$

Then

$$\mathrm{d}^{0,1}\left(\Sigma, P^{-1}(0)\right) \leq C\sqrt{\varepsilon}. \tag{5.47}$$

Proof. Let all notation and hypotheses hold. Let $x \in \Sigma \cap B(0,1)$. Write $x = (re, x_4) \in \mathbb{R}^3 \times \mathbb{R}$, for $r \geq 0$ and $e = (e_1, e_2, e_3) \in \mathbb{R}^3$, $|e| = 1$. We consider two cases of (5.46). First, we suppose that

$$0 \leq P(x) \leq \varepsilon. \tag{5.48}$$

By (5.45), η_i is negative for $i \in \{1, 2, 3\}$. Thus by (5.48), there exists $\hat{r} \geq r$ such that $\hat{x} := (\hat{r}e, x_4)$ satisfies

$$P(\hat{x}) = 0. \tag{5.49}$$

We next note that

$$|x - \hat{x}| = |r - \hat{r}|. \tag{5.50}$$

Let $\eta_e = \sum_{i=1}^{3} \eta_i e_i^2$. Because $|e| = 1$ and (5.45), we get that $|\eta_e + 1/2| \leq 1/8$. We compute

$$\varepsilon \geq |P(x) - P(\hat{x})| = \left| \sum_{i=1}^{3} \eta_i e_i^2 (r^2 - \hat{r}^2) \right| = |r^2 - \hat{r}^2||\eta_e|. \tag{5.51}$$

We recall that $\hat{r} \geq r \geq 0$. We note that if $a, b > 0$, then $|a - b| \leq \sqrt{|a^2 - b^2|}$. We use these observations and (5.51) to compute

$$|\hat{r} - r| \leq \sqrt{|\hat{r}^2 - r^2|} \leq \frac{1}{\sqrt{|\eta_e|}}\sqrt{\varepsilon} \leq C\sqrt{\varepsilon}. \tag{5.52}$$

We now consider the other case of (5.46). Suppose that

$$-\varepsilon \leq P(x) \leq 0. \tag{5.53}$$

For ease, assume $x_4 \geq 0$ (the case $x_4 \leq 0$ will be similar). Because $\eta_4 > 0$ by (5.45), $x_4 \geq 0$, and (5.53), there exists $\hat{x_4} \geq x_4$ such that $\hat{x} := (x_1, x_2, x_3, \hat{x_4})$ satisfies

$$P(\hat{x}) = 0. \tag{5.54}$$

Note that

$$|x - \hat{x}| = |x_4 - \hat{x_4}|. \tag{5.55}$$

Note also that

$$\varepsilon \geq |P(x) - P(\hat{x})| = |\eta_4(x_4^2 - \hat{x_4}^2)| = |\eta_4||x_4^2 - \hat{x_4}^2|. \tag{5.56}$$

As before, we note that $\hat{x_4} \geq x_4 \geq 0$. Hence, from (5.45) and (5.56), we get that

$$|\hat{x_4} - x_4| \leq \sqrt{|\hat{x_4}^2 - x_4^2|} \leq \frac{1}{\sqrt{\eta_4}}\sqrt{\varepsilon} \leq C\sqrt{\varepsilon}. \tag{5.57}$$

Hence, we've shown that for any $x \in \Sigma \cap B(0,1)$, there exists $\hat{x} \in P^{-1}(0)$ such that $|x - \hat{x}| \leq C\sqrt{\varepsilon}$. Hence,

$$\widetilde{\mathrm{d}}^{0,1}\left(\Sigma, P^{-1}(0)\right) \leq C\sqrt{\varepsilon}. \tag{5.58}$$

Because P is a homogenous polynomial, $P^{-1}(0)$ is a cone centered at 0. Hence, we apply Lemma A.1.2 to (5.58) to get

$$\mathrm{d}^{0,1}\left(\Sigma, P^{-1}(0)\right) \leq C\sqrt{\varepsilon}. \tag{5.59}$$

□

5.2.2 The Tangent Cone at the Singularity

In this section we show that at a nonflat point there is a unique KP cone C based at the origin to which the blowups converge at a Hölder rate. Specifically, we strive to show that there are an $r_0 > 0$ small enough, constant C, and exponent $\beta_1 = \beta_1(\alpha)$ such that for $0 < r \leq r_0$,

$$\mathrm{D}^{0,r}\left[\mathrm{C}, \Sigma\right] \leq Cr^{\beta_1}.$$

To do this, we will apply Lemmas 5.2.2(6), 5.2.5, and 5.2.6. Lemma 5.2.5 allows us to diagonalize $Q_{0,r}$, which allows us to apply Lemma 5.2.6 with the bounds from Lemma 5.2.2(6). Taken together, these will give us the estimates on $\mathrm{d}^{0,r}(\Sigma, \mathrm{C})$. To get estimates on $\mathrm{d}^{0,r}(\mathrm{C}, \Sigma)$, we will apply Theorem A.3.4, which will allow us to show that every point in $\mathrm{C} \cap B(0,r)$ has a point in Σ which is no further than $\mathrm{d}^{0,2r}(\Sigma, \mathrm{C})$, and this will finish the argument.

Fix $\varepsilon > 0$ small. By Theorem 4.3.4, there is some radius r_0 small enough such that

$$\Theta^{\mathcal{C}}_{\Sigma}(x,r) < \varepsilon \quad \text{for } x \in \Sigma \cap B(0,r_0), 0 < r \leq r_0. \tag{5.60}$$

We work at a scale of $\rho = r^{1+\gamma_0}$ for some $0 < \gamma_0 < \alpha/2$. For ease, let $\rho_0 = r_0^{1+\gamma_0}$. Define $\mathcal{E}(\rho)$ by

$$\mathcal{E}(\rho) = \widetilde{Q}^{-1}_{0,r}(0). \tag{5.61}$$

We recall Lemma 5.2.2(6), which tells us that that for $r_0 < 1/4$, $0 < r \leq r_0$,

$$\widetilde{Q}_{0,\tau r}(z) \leq C r^{\beta_0} \quad \text{for all } z \in \Sigma_\rho \cap B(0,2) \text{ and } \tau \in [1/2, 1]. \tag{5.62}$$

From Lemma 5.2.5, we know that for r_0 small enough, we can diagonalize $\widetilde{Q}_{0,r}$ as $\widetilde{Q}_{0,r} = \sum_{i=1}^4 (\lambda_i - 1)x_i^2$ with $|\lambda_i - 1/2| < 1/8$ for $i = 1,2,3$ and $\lambda_4 - 3/2| < 1/8$. This allows us to apply Lemma 5.2.6 with the bound of (5.62), which yields that

$$\mathrm{d}^{0,2}\left(\Sigma_\rho, \mathcal{E}((\tau r)^{1+\gamma_0})\right) \leq C r^{\frac{\beta_0}{2}} \quad \text{for all } \tau \in [1/2, 1]. \tag{5.63}$$

Manipulating the τ above,

$$\mathrm{d}^{0,2}\left(\Sigma_\rho, \mathcal{E}(\tau\rho)\right) \leq C r^{\frac{\beta_0}{2}} \quad \text{for all } \tau \in \left[1/2^{1+\gamma_0}, 1\right]. \tag{5.64}$$

In particular, the above holds for all $\tau \in [1/2, 1]$.

Let $\sigma > 0$. By Theorem 4.3.4, for ρ_0 small enough and $0 < \rho \leq \rho_0$, there exists a KP cone $\mathrm{C}(\rho)$ based at the origin such that

$$\mathrm{D}^{0,2}\left[\Sigma_\rho, \mathrm{C}(\rho)\right] < \sigma. \tag{5.65}$$

From (5.64) and (5.65), it follows that

$$\mathrm{d}^{0,2}\left(C(\rho), \mathcal{E}(\tau\rho)\right) < \sigma + C r^{\frac{\beta_0}{2}} \quad \text{for all } 0 < \rho \leq \rho_0, \tau \in [1/2, 1]. \tag{5.66}$$

Fix $\tau \in [1/2, 1]$. Let $a \in \mathcal{E}(\tau\rho)$ such that$1/2 \leq |a| \leq 1$. Let ν_a be the normal vector to $\mathcal{E}(\tau\rho)$ at a, and let $\ell_a = \{a + t\nu_a \mid t \in \mathbb{R}\}$ be the line going through a parallel to ν_a.

Let $\delta > 0$. Recall that $\mathcal{E}(\tau\rho)$ is the zero set of a homogenous degree 2 polynomial. By (5.66) and requiring that σ and ρ_0 be small enough, we can not only guarantee that ℓ_a interesects $\mathrm{C}(\rho)$, but also that at the point of intersection (nearest to a), the angle between ℓ_a and $\mathrm{C}(\rho)$ is arbitrarily close to $\pi/2$. In particular, we can guarantee that it is greater than $\pi/4$. It follows from this observation, (5.60), and (5.65) that we may invoke Theorem A.3.4 with ν_a at the point of intersection. Hence, for small enough ρ_0 and σ, there exists $z \in \Sigma_\rho \cap \ell_a$ such that $\mathrm{dist}(z, \mathrm{C}(\rho)) \leq \delta$. Coupling this with (5.66), we get that

$$\mathrm{dist}(z, \mathcal{E}(\tau\rho)) \leq \delta + 2\sigma + Cr^{\frac{\beta_0}{2}}. \tag{5.67}$$

For δ, σ, and ρ_0 small enough, the fact that ν_a is normal to $\mathcal{E}(\tau\rho)$ implies that the nearest point on $\mathcal{E}(\tau\rho)$ to z is a. Hence,

$$|z - a| = \mathrm{dist}(z, \mathcal{E}(\tau\rho)) \leq 2\,\mathrm{d}^{0,2}\left(\Sigma_\rho, \mathcal{E}(\tau\rho)\right) \leq Cr^{\frac{\beta_0}{2}}. \tag{5.68}$$

Recall that our only assumption on a was that $1/2 \leq |a| \leq 1$ and $a \in \mathcal{E}(\tau\rho)$. Hence, (5.68) tells us that

$$\mathrm{d}^{0,1}\left(\mathcal{E}(\tau\rho) \setminus B\left(0, 1/2\right), \Sigma_\rho\right) \leq Cr^{\frac{\beta_0}{2}}. \tag{5.69}$$

Hence, combining (5.64) and (5.69), we get that

$$\widetilde{\mathrm{d}}^{0,1}\left(\mathcal{E}(\tau\rho) \setminus B\left(0, 1/2\right), \mathcal{E}(\tau'\rho)\right) \leq Cr^{\frac{\beta_0}{2}} \quad \text{ for } \tau, \tau' \in [1/2, 1]\,. \tag{5.70}$$

From the fact that both $\mathcal{E}(\tau\rho)$ and $\mathcal{E}(\tau'\rho)$ are cones based at 0, it follows from (5.70) that

$$\mathrm{d}^{0,1}\left(\mathcal{E}(\tau\rho), \mathcal{E}(\tau'\rho)\right) \leq Cr^{\frac{\beta_0}{2}} \quad \text{ for } \tau, \tau' \in [1/2, 1] \tag{5.71}$$

(see Lemma A.1.2). Because condition (5.71) is symmetric, we get that

$$\mathrm{D}^{0,1}\left[\mathcal{E}(\tau\rho), \mathcal{E}(\tau'\rho)\right] \leq Cr^{\frac{\beta_0}{2}} \quad \text{ for } \tau, \tau' \in [1/2, 1]\,. \tag{5.72}$$

We now exploit (5.72) to get a rate of convergence for the one parameter family $\mathcal{E}(\rho)$. Suppose that $0 < \rho' < \rho \leq \rho_0$. Write $\rho' = \tau\rho/2^N$ for $\tau \in [1/2, 1]$, $N \in \mathbb{Z}_{\geq 0}$. Then we get from (5.72) that

$$\begin{aligned} \mathrm{D}^{0,1}\left[\mathcal{E}(\rho),\mathcal{E}(\rho')\right] &\leq \sum_{j=0}^{N-1} \mathrm{D}^{0,1}\left[\mathcal{E}\left(\frac{\rho}{2^j}\right),\mathcal{E}\left(\frac{\rho}{2^{j+1}}\right)\right] + \mathrm{D}^{0,1}\left[\mathcal{E}\left(\frac{\rho}{2^N}\right),\mathcal{E}(\rho')\right] \\ &\leq \sum_{j=0}^{N} C\left(\frac{r}{2^j}\right)^{\frac{\beta_0}{2}} \leq Cr^{\frac{\beta_0}{2}} \sum_{j=0}^{\infty}\left(\frac{1}{2^{\frac{\beta_0}{2}}}\right)^j = Cr^{\frac{\beta_0}{2}}. \end{aligned} \tag{5.73}$$

We now recall Lemma 5.2.5 and that $\mathcal{E}(\rho) = \widetilde{Q}_{0,r}^{-1}(0)$. Together, they imply that any convergent sequence $\mathcal{E}(\rho_i)$ is converging to a KP cone based at the origin. Moreover, by (5.73), the one parameter family is Cauchy and hence converges to a unique KP cone based at the origin. Call this KP cone C. We now use (5.73) to compute that for $0 < \rho \leq \rho_0$,

$$\mathrm{D}^{0,1}\left[\mathcal{E}(\rho),\mathrm{C}\right] \leq \lim_{\rho'\downarrow 0} \mathrm{D}^{0,1}\left[\mathcal{E}(\rho),\mathcal{E}(\rho')\right] + \mathrm{D}^{0,1}\left[\mathcal{E}(\rho'),\mathrm{C}\right] \leq Cr^{\frac{\beta_0}{2}}. \tag{5.74}$$

We now wish to show that

$$\mathrm{D}^{0,1}\left[\Sigma_\rho,\mathrm{C}\right] \leq Cr^{\frac{\beta_0}{2}}. \tag{5.75}$$

First, we note that by (5.64), Lemma A.1.2, and (5.74), we get that

$$\mathrm{d}^{0,1}\left(\Sigma_\rho,\mathrm{C}\right) \leq \mathrm{d}^{0,1}\left(\Sigma_\rho,\mathcal{E}(\rho)\right) + \mathrm{d}^{0,1}\left(\mathcal{E}(\rho),\mathrm{C}\right) \leq Cr^{\frac{\beta_0}{2}}. \tag{5.76}$$

We now recall that by Theorem 4.3.4, we have that for any sequence $\rho_i \downarrow 0$ with Σ_{ρ_i} convergent, $\Sigma_{\rho_i} \to \mathrm{C}'$ for some KP cone C'. We note that (5.76) implies that $\mathrm{C}' \subseteq \mathrm{C}$, and because each is a KP cone, $\mathrm{C}' = \mathrm{C}$. It follows that $\mathrm{D}^{0,1}\left[\Sigma_{\rho_i},\mathrm{C}\right] \to 0$ as $i \to \infty$ for any convergent sequence ρ_i. From this observation, we get that

$$\mathrm{D}^{0,1}\left[\Sigma_\rho,\mathrm{C}\right] \to 0 \quad \text{ as } \rho \downarrow 0. \tag{5.77}$$

Let $0 < \rho \leq \rho_0$ and $a \in \mathrm{C} \cap B(0,1) \setminus B(0,1/2)$. Similarly to how we proved (5.69), we apply (5.60), (5.76), (5.77) to apply Theorem A.3.4 to the vector ν_a (which we recall is the unit normal to C at a). This gives us that there is a point $z \in \Sigma_\rho$ with $z = a + t\nu_a$ and t small. As before, we get that $|z-a| = \operatorname{dist}(z,\mathrm{C}) \leq |a|\,\mathrm{d}^{0,|a|}\left(\Sigma,\mathrm{C}\right) \leq Cr^{\frac{\beta_0}{2}}$. Hence, we get that

$$\mathrm{d}^{0,1}\left(\mathrm{C} \setminus B(0,1/2),\Sigma_\rho\right) \leq Cr^{\frac{\beta_0}{2}}. \tag{5.78}$$

Let $a \in \mathrm{C} \cap B(0,1) \setminus \{0\}$. Then we apply scale invariance of C and (5.78) to compute that

$$\mathrm{d}^{0,|a|}\left(\mathrm{C} \setminus B(0,|a|/2),\Sigma_\rho\right) = \mathrm{d}^{0,1}\left(\mathrm{C} \setminus B(0,1/2),\Sigma_{|a|\rho}\right) \leq Cr^{\frac{\beta_0}{2}}. \tag{5.79}$$

Hence,

$$\operatorname{dist}(a, \Sigma_\rho) \le C|a| r^{\frac{\beta_0}{2}} \le C r^{\frac{\beta_0}{2}}. \tag{5.80}$$

From which we get

$$\mathrm{d}^{0,1}\left(\mathrm{C}, \Sigma_\rho\right) \le C r^{\frac{\beta_0}{2}}. \tag{5.81}$$

Combining (5.76) and (5.81),

$$\mathrm{D}^{0,1}\left[\Sigma_\rho, \mathrm{C}\right] \le C r^{\frac{\beta_0}{2}}. \tag{5.82}$$

It is now convenient to switch back from $\rho = r^{1+\gamma_0}$ to r. Making the substitution to (5.78), we get

$$\mathrm{D}^{0,1}\left[\Sigma_r, \mathrm{C}\right] \le C r^{\frac{\beta_0}{2(1+\gamma_0)}}. \tag{5.83}$$

Let $\beta_1 = \beta_0/(2(1+\gamma_0))$. Recall that β_0 and γ_0 depended only on α, and hence so does β_1. We have shown the following.

Lemma 5.2.7. *Recall hypotheses (5.6) and (5.60). There is a (unique) KP cone* C *based at the origin,* $r_0 = r_0(C_0, \alpha, \varepsilon) > 0$ *small enough, a constant* $C = C(C_0, \alpha)$*, and* $\beta_1 = \beta_1(\alpha)$ *(defined above) such that for* $0 < r \le r_0$*,*

$$\mathrm{D}^{0,1}\left[\Sigma_r, \mathrm{C}\right] \le C r^{\beta_1}. \tag{5.84}$$

5.2.3 Hölder Reifenberg Flatness Away from the Origin

In this section we investigate how the quantity $\Theta^{\mathcal{G}}_{\Sigma}(x, r)$ behaves for points x near the origin and scales r which are appropriately small in terms of $|x|$. Recall hypotheses (5.6). We begin by stating a result of [9].

Theorem 5.2.8. *Suppose that* μ *is a Radon measure on* $\mathbb{R}^n$ *which is* $(\alpha, n-1)$*-Hölder asymptotically optimally doubling with convergence constants* C_K *for each compact set* $K \subseteq \Sigma$*. Then there exist* $\delta = \delta(n, \alpha)$ *and* $r_1 = r_1(C_K, \alpha, n, \delta)$ *such that if* $r_1' \le r_1$ *and*

$$\sup_{x \in K, 0 < r \le r_1'} \Theta^{\mathcal{G}}_{\Sigma}(x, r) \le \delta,$$

then there exist $C_K' = C_K'(C_K, \delta, n, \alpha)$ *(but not otherwise dependent on* μ*) and* $\beta = \beta(\alpha)$ *such that for all* $0 < r \le r_1'$ *and* $x \in K$*,*

$$\Theta^{\mathcal{G}}_{\Sigma}(x, r) \le C_K' \left(\frac{r}{r_1'}\right)^{\beta}.$$

Remark 5.2.9. *Although this result is slightly more quantitative than [9] Proposition 8.6, it is not difficult to obtain. The result is proven in Section 8 of [9], and the only place where a more complicated relationship between δ and r_1 can arise is in the proof of Lemma 8.2. However, one removes all ambiguity of the dependence in this proof by simply choosing r_0 small enough such that $9\varepsilon(\gamma_1) \leq 1/(8n+8)$ and δ small enough such that $9C\delta \leq 1/(8n+8)$.*

In light of Theorem 5.2.8, we recall Corollary A.3.3, which tells us that there is an A large enough such that for some $r_0 > 0$,

$$\Theta_\Sigma^{\mathcal{G}}(x,r) < \delta \quad \text{for all } x \in \Sigma \cap B(0,r_0) \setminus \{0\}, 0 < r \leq \frac{|x|}{A}. \tag{5.85}$$

Hence we can apply Theorem 5.2.8 to get

$$\Theta_\Sigma^{\mathcal{G}}(x,r) \leq C\left(\frac{r}{r_1'}\right)^\beta \quad \text{for all } x \in \Sigma \cap B(0,r_0) \setminus \{0\}, 0 < r \leq r_1' \leq \frac{|x|}{A}. \tag{5.86}$$

Taking

$$r_1' = \frac{1}{A}|x| \tag{5.87}$$

gives that

$$\Theta_\Sigma^{\mathcal{G}}(x,r) \leq C\left(\frac{r}{|x|}\right)^\beta. \tag{5.88}$$

We now fix $0 < \gamma_1 < \beta_1$. Let $0 < r \leq |x|^{1+\gamma_1}/A$, we get that $1/|x| \leq Cr^{\frac{1}{1+\gamma_1}}$. Combining this with (5.88) gives us that

$$\Theta_\Sigma^{\mathcal{G}}(x,r) \leq C\left(\frac{r}{r^{\frac{1}{1+\gamma_1}}}\right)^\beta = Cr^{\frac{\beta\gamma_1}{1+\gamma_1}}. \tag{5.89}$$

Setting $\beta_2 = \frac{\gamma_1\beta}{1+\gamma_1}$, (5.89) says that

$$\Theta_\Sigma^{\mathcal{G}}(x,r) \leq Cr^{\beta_2}. \tag{5.90}$$

Hence, we have proven the following.

Lemma 5.2.10. *Recall hypotheses (5.6). There exist a constant C and an $r_0 > 0$ small enough such that for all $0 < r \leq r_0$,*

$$\Theta_\Sigma^{\mathcal{G}}(x,r) \leq Cr^{\beta_2}. \tag{5.91}$$

5.3 *Parametrization*

In this section, we use the geometric information we have gathered to construct a $C^{1,\beta}$ parametrization of a neighborhood of 0 by a KP cone. We work toward the Theorem 5.3.1, which when paired with Lemmas 5.2.7 and 5.2.10, proves Theorem 5.2.1.

Theorem 5.3.1. *Let $\Sigma \subseteq \mathbb{R}^4$ be a closed set containing 0, $0 < \gamma < \beta_1$, and $0 < \beta_2$. There exists $\beta = \beta(\gamma, \beta_1, \beta_2)$ with the following property. Let C be a KP cone cnetered at 0, and assume the following estimates on Σ:*

(E0) for $x \in B(0, r_0) \cap \Sigma$ and $0 < r \leq 2r_0$, $\Theta^{\mathcal{C}}_{\Sigma}(x, r) < \varepsilon$,

(E1) for $0 < r \leq 2r_0$, $\mathrm{D}^{0,r}\left[\Sigma, \mathrm{C}\right] \leq \min(\sigma, C_1 r^{\beta_1})$,

(E2) for $x \in \Sigma \cap B(0, r_0) \setminus \{0\}$, $0 < r \leq 16\frac{|x|^{1+\gamma}}{A}$, $\Theta^{\mathcal{G}}_{\Sigma}(x, r) \leq \min(\delta, C_2 r^{\beta_2})$.

For $A > 1$ large enough and $\sigma, \delta, r_0 > 0$ small enough, we have that Σ admits a $C^{1,\beta}$ parametrization by C. That is, there exist neighborhoods U of 0 and U' of 0 and a diffeomorphism $\varphi \in C^{1,\beta}(U \to U')$ such that $\varphi(\mathrm{C} \cap U) = \Sigma \cap U'$. Further, φ has the property that $\varphi(0) = 0$ and $D_0\varphi = \mathrm{Id}$, and U has the property that $U \cap \mathrm{C} \supseteq B(0, r_0) \cap \mathrm{C}$.

Let $\Sigma \subseteq \mathbb{R}^4$ be a closed set containing 0. We fix exponents $0 < \gamma < \beta_1$, and $0 < \beta_2$. The parameters $\varepsilon, \sigma, \delta, 1/A$ and r_0 will be chosen small enough throughout this section. Let (x_1, x_2, x_3, x_4) be orthonormal coordinates centered at the origin and $\mathrm{C} = \{x_4^2 = x_1^2 + x_2^2 + x_3^2\}$.

For $x \in \Sigma \cap B(0, r_0), \setminus\{0\}$ and $0 < r \leq |x|^{1+\gamma}/A$, let $P(x, r)$ be a plane such that

$$\mathrm{D}^{x,r}\left[P(x, r), \Sigma\right] \leq \min(\delta, Cr^{\beta_2}) \quad \text{ and } x \in P(x, r). \tag{5.92}$$

For each cross section $\mathrm{C}_h = \mathrm{C} \cap \{x_4 = h\}$, we note that C_h is the 2 sphere of radius h centered at $(0, h)$ inside of the plane $\{x_4 = h\}$. This means that nearest point projection in the cross section is defined for all $x \neq (0, h)$. Let $\tau(x) : \mathbb{R}^4 \to \mathbb{R}^3$ be orthogonal projection onto the first 3 coordinates. We thus define $\pi : \mathbb{R}^4 \setminus \{x : \tau(x) = 0\} \to \mathrm{C}$ to be nearest point projection in the cross section. One can check that

$$\pi(x) = \left(\frac{|x_4|}{|\tau(x)|}\tau(x), x_4\right). \tag{5.93}$$

We define the vector field η on $\mathbb{R}^4 \setminus \mathbb{R}x_4$ by

$$\eta_x = \frac{1}{|\tau(x)|}\,(-\tau(x), 0)\,. \tag{5.94}$$

Note that for $x = a \in \mathrm{C}$, this is the vector normal to the cross section C_{a_4} at a viewed in the plane $\{x_4 = a_4\}$. Hence

$$\pi(x) - x = \pm\,\mathrm{dist}(x, \mathrm{C}_{x_4})\eta_{\pi(x)} \tag{5.95}$$

depending on whether x is inside or outside of the sphere C_{a_4}. We also note that $\pi(x)$ is the nearest point in C to x along the line based at x in the direction η_x. Thus, for $a \in \mathbb{R}^4 \setminus \mathbb{R}x_4$, we define the half line

$$\ell_a = \{b \in \mathbb{R}^4 : b - a = t\eta_a \text{ for some } t \in \mathbb{R} \text{ and } \tau(a)\cdot\tau(b) \geq 0\}. \tag{5.96}$$

Note that the half lines ℓ_a are the integral curves of η and that for any $x \in \mathbb{R}^4 - \mathbb{R}x_4$, $\pi(x)$ is the unique intersection of the integral curve containing x with C. For $a \in \mathrm{C}$, note that η_a is the normal vector to the cross section C_{a_4}, as opposed to the normal vector to the cone C, which is

$$\nu_a = \frac{1}{|a|}\,(-\tau(a), a_4)\,. \tag{5.97}$$

Further, we note the following.

Lemma 5.3.2. *Recall hypotheses (E0)-(E2). For $x \in \mathbb{R}^4 \setminus \mathbb{R}x_4$, $|x - \pi(x)| = \sec(\pi/4)\,\mathrm{dist}(x, \mathrm{C})$.*

Proof. First, we note that because C is a smooth manifold away from 0 and 0 is not the closest point to x, the distance $\mathrm{dist}(x, \mathrm{C})$ is the length of the vector based at x in the $\nu_{\pi(x)}$ direction ending on C. Second, we note that the angle between $\nu_{\pi(x)}$ and $\eta_{\pi(x)}$ is $\frac{\pi}{4}$. Thus because C is a cone, the vector based at x pointing in the η_x ending on C has length $\sec(\frac{\pi}{4})\,\mathrm{dist}(x, \mathrm{C})$. □

Lemma 5.3.3. *Recall hypotheses (E0)-(E2). For $x \in \Sigma \cap B(0, r_0) \setminus \{0\}$,*

$$|\pi(x) - x| \leq \min(C|x|^{1+\beta_1}, 2\sigma|x|). \tag{5.98}$$

Proof. Let $r = 2|x|$. Applying Lemma 5.3.2 and assumption (E1), we get that

$$\mathrm{dist}(x, \mathrm{C}) \leq \min(C|x|^{1+\beta_1}, 2\sigma|x|). \tag{5.99}$$

We then apply Lemma 5.3.2 and conclude. □

Lemma 5.3.4. *Recall hypotheses (E0)-(E2) and (5.92). For r_0 small enough, depending on C_2, β_2 and γ, we have the following. Let $x_1, x_2 \in \Sigma \cap B(x, r_0), 0 < t_1, t_2 < r_0$ such that*

$$\frac{1}{4}t_1 \leq \frac{1}{2}t_2 \leq t_1 \leq \frac{16}{A}|x_1|^{1+\gamma} \quad \textit{and} \quad |x_1 - x_2| \leq \frac{t_1}{16}. \tag{5.100}$$

Then $\measuredangle(P(x_1, t_1), P(x_2, t_2)) \leq Ct_1^{\beta_2}$.

Proof. By Lemma A.2.1, it will suffice to find a radius r such that

$$\mathrm{d}^{x_1,r}\left(P(x_1, t_1), P(x_2, t_2)\right) \leq Ct_1^{\beta_2}. \tag{5.101}$$

We take r to be $t_1/8$. Let $x \in P_1 \cap B(x_1, t_1/8)$. By (5.92) there exists $y \in \Sigma \cap B(x_1, t_1)$ with $|x - y| \leq Ct_1^{1+\beta_2}$. Further, we may take $|x - y| \leq \delta t_1 \leq t_1/8$. We estimate that

$$|y - x_2| \leq |y - x| + |x - x_1| + |x_1 - x_2| \leq 5\frac{t_1}{16} < \frac{t_1}{2} \leq t_2. \tag{5.102}$$

Hence, $y \in B(x_2, t_2)$. So by assumption there exists $z \in P(x_2, t_2) \cap B(x_2, t_2)$ such that

$$|y - z| \leq Ct_2^{1+\beta_2} \leq Ct_1^{1+\beta_2}.$$

Thus,

$$|x - z| \leq |x - y| + |y - z| \leq Ct_1^{1+\beta_2}. \tag{5.103}$$

Because for each $x \in P(x_1, t_1) \cap B(x_1, t_1/8)$ there exists $z \in P(x_2, t_2)$ such that (5.103) holds, we have that

$$\widetilde{\mathrm{d}}^{x_1,\frac{t_1}{8}}\left(P(x_1, t_1), P(x_2, t_2)\right) \leq Ct_1^{\beta_2}. \tag{5.104}$$

Because P_1 is a cone based at x_1, by Lemma A.1.2 we obtain that (5.101) holds for $r = t_1/8$, and we are done. □

Corollary 5.3.5. *Recall hypotheses (E0)-(E2). For $x, y \in B(0, r_0)$, the following hold:*

(1) $\measuredangle(P(x, r), P(x, r')) \leq Cr^{\beta_2}$ for $0 < r' \leq r \leq 16|x|^{1+\gamma}/A$

(2) $P(x) = \lim_{r\downarrow 0} P(x,r)$ *exists*

(3) $\measuredangle\left(P\left(x,r\right), P(x)\right) \leq Cr^{\beta_2}$ *for* $0 < r \leq |x|^{1+\gamma}/A$

(4) $\measuredangle\left(P(x), P(y)\right) \leq C|x-y|^{\beta_2}$ *when* $|x-y| \leq |x|^{1+\gamma}/A$

(5) $\operatorname{dist}(y, P(x)) \leq C|x-y|^{1+\beta_2}$ *when* $|x-y| \leq |x|^{1+\gamma}/A$

Proof. Let $B \in [1/2, 1)$ and write $r' = Br/2^j$ for $j \in \mathbb{N}$. By Lemma A.2.1, Lemma 5.3.4, and subadditivity of angles we have that

$$\begin{aligned}\measuredangle(P(x,r), P(x,r')) &\leq \sum_{i=0}^{j-1} \measuredangle\left(P\left(x, r/2^i\right), P\left(x, r/2^{i+1}\right)\right) + \measuredangle\left(P\left(x, r/2^j\right), P\left(x, Br/2^j\right)\right) \\ &\leq C\sum_{i=0}^{j} \left(r/2^i\right)^{\beta_2} \leq Cr^{\beta_2} \sum_{i=0}^{\infty} \left(1/2^{\beta_2}\right)^i = Cr^{\beta_2},\end{aligned} \tag{5.105}$$

establishing (1). In particular, it follows that the one parameter family $P(x,r)$ is Cauchy as $r \downarrow 0$, establishing (2). With the help of (1) and subadditivity of angles, we compute

$$\begin{aligned}\measuredangle(P(x,r), P(x)) &\leq \lim_{r'\downarrow 0} \measuredangle(P(x,r), P(x,r')) + \measuredangle(P(x,r'), P(x)) \\ &\leq Cr^{\beta_2} + \lim_{r'\downarrow 0} \measuredangle(P(x,r'), P(x)) = Cr^{\beta_2},\end{aligned} \tag{5.106}$$

establishing (3).

We now prove (4). Let $x \in B(0, r_0) \cap \Sigma \setminus \{0\}$ and $|x-y| \leq |x|^{1+\gamma}/A$. Let $r = 16|x-y|$. Then we apply (3), Lemma 5.3.4, and subadditivity of angles to compute

$$\begin{aligned}\measuredangle(P(x), P(y)) &\leq \measuredangle(P(x), P(x,r)) + \measuredangle(P(x,r), P(y,r)) + \measuredangle(P(y,r), P(y)) \\ &\leq Cr^{\beta_2} = C|x-y|^{\beta_2}.\end{aligned} \tag{5.107}$$

Finally, we prove (5). Under the same assumptions that $x \in B(0, r_0) \cap \Sigma \setminus \{0\}$ and $|x-y| \leq |x|^{1+\gamma}/A$, we apply (5.92) and (3) to get that

$$\begin{aligned}\operatorname{dist}(y, P(x)) &\leq \operatorname{dist}(y, P(x, 2|x-y|)) + e(P(x) \cap B(x, 2|x-y|), P(x, 2|x-y|) \cap B(x, 2|x-y|)) \\ &\leq \operatorname{dist}(y, P(x, 2|x-y|)) + 2|x-y| \operatorname{D}^{x, 2|x-y|}\left[P(x), P(x, 2|x-y|)\right] \\ &\leq C|x-y|^{1+\beta_2}.\end{aligned} \tag{5.108}$$

□

Lemma 5.3.6. *Recall hypotheses (E0)-(E2). Let* $x \in \Sigma \cap B(0, r_0)$*,* $R = 8|x|^{1+\gamma}/A$*, and* $0 < r \leq 2R$*. Then*

$$\mathrm{d}^{x,R}\left(\mathrm{C}, P(x,r)\right) \leq C|x|^{\min(\beta_1-\gamma, \beta_2(1+\gamma))}. \tag{5.109}$$

Proof. Let $a \in \mathrm{C} \cap B(x,R)$. Note that $B(x,R) \subseteq B(0, 2|x|)$. Hence by (E1) there exists $y \in \Sigma \cap B(0, 2|x|)$ such that

$$|a-y| \leq C|x|^{1+\beta_1} = R\left(C|x|^{\beta_1-\gamma}\right). \tag{5.110}$$

Further, we have that

$$|x-y| \leq |x-a| + |a-y| \leq R + R\left(C|x|^{\beta_1-\gamma}\right) \leq R\left(1 + C|r_0|^{\beta_1-\gamma}\right) \leq 2R, \tag{5.111}$$

for r_0 small enough. Hence, $y \in B(x, 2R) \cap \Sigma$. Because $2R = 16|x|^{1+\gamma}/A$, (E2) tells us that there exists $p \in P(x, 2R)$ such that

$$|y-p| \leq 2R \cdot CR^{\beta_2} = R \cdot C|x|^{\beta_2(1+\gamma)}. \tag{5.112}$$

Putting (5.110) and (5.112) together, we get

$$|a-p| \leq |a-y| + |y-p| \leq CR\left(|x|^{\beta_1-\gamma} + |x|^{\beta_2(1+\gamma)}\right). \tag{5.113}$$

Because for each $a \in B(x,R)$ there exists a $p \in P(x, 2R)$ such that (5.113) is satisfied, we get that

$$\widetilde{\mathrm{d}}^{x,R}\left(\mathrm{C}, P(x,2R)\right) \leq C|x|^{\min(\beta_1-\gamma, \beta_2(1+\gamma))}. \tag{5.114}$$

Thus, by Lemma A.1.2, (5.114) tells us that

$$\mathrm{d}^{x,R}\left(\mathrm{C}, P(x,2R)\right) \leq C|x|^{\min(\beta_1-\gamma, \beta_2(1+\gamma))}. \tag{5.115}$$

Corollary 5.3.5, (A.17), and (5.115) tells us that

$$\mathrm{d}^{x,R}\left(\mathrm{C}, P(x,r)\right) \leq \mathrm{d}^{x,R}\left(\mathrm{C}, P(x,2R)\right) + \mathrm{d}^{x,R}\left(P(x,2R), P(x,r)\right) \leq C|x|^{\min(\beta_1-\gamma, \beta_2(1+\gamma))}. \tag{5.116}$$

□

We now derive information about the angle $P(x)$ makes with the tangent plane $T_{\pi(x)}\mathrm{C}$. We define

$$\beta_3 = \min(\beta_1 - \gamma, \beta_2(1+\gamma), \gamma). \tag{5.117}$$

Lemma 5.3.7. *Recall hypotheses (E0)-(E2). For $x \in \Sigma \cap B(0, r_0)$, $\measuredangle\left(P(x), T_{\pi(x)}\mathrm{C}\right) \leq C|x|^{\beta_3}$.*

Proof. Let $R = 8|x|^{1+\gamma}/A$. By Lemma 5.3.6, we have that

$$\mathrm{d}^{x,R}\left(\mathrm{C}, P(x,r)\right) \leq C|x|^{\min(\beta_1-\gamma,\beta_2(1+\gamma)}. \tag{5.118}$$

Letting $r \downarrow 0$, we get

$$\mathrm{d}^{x,R}\left(\mathrm{C}, P(x)\right) \leq C|x|^{\min(\beta_1-\gamma,\beta_2(1+\gamma))}. \tag{5.119}$$

We apply Lemma 5.3.3 to get

$$|\pi(x) - x| \leq C|x|^{1+\beta_1} \leq 8\frac{|x|^{1+\gamma}}{A}\left(Cr_0^{\beta_1-\gamma}\right) = R\left(Cr_0^{\beta_1-\gamma}\right), \tag{5.120}$$

and hence $|x|$ and $|\pi(x)|$ are within a factor of 2 for r_0 small enough. We use this observation and Lemma 4.3.1 to see that

$$\mathrm{d}^{\pi(x),\frac{R}{2}}\left(T_{\pi(x)}\mathrm{C}, \mathrm{C}\right) \leq C\frac{R}{|x|} = C|x|^{\gamma}. \tag{5.121}$$

Let $p \in T_{\pi(x)}\mathrm{C} \cap B(\pi(x), R/2)$. Then (5.121) says there exists $c \in \mathrm{C} \cap B(\pi(x), R/2)$ such that

$$|p - c| \leq CR|x|^{\gamma}. \tag{5.122}$$

Next, we claim that for r_0 small enough, $c \in B(x, R)$. For r_0 small enough, (5.120) tells us that $|\pi(x) - x| \leq R/2$. Hence, $c \in B(\pi(x), R/2) \subseteq B(x, R)$. By (5.119), there exists $q \in P(x)$ such that

$$|c - q| \leq CR|x|^{\min(\beta_1-\gamma,\beta_2(1+\gamma))}. \tag{5.123}$$

From (5.122) and (5.123), we get that

$$|p - q| \leq CR|x|^{\min(\beta_1-\gamma,\beta_2(1+\gamma))} + CR|x|^{\gamma} \leq CR|x|^{\beta_3} \tag{5.124}$$

(recall (5.117)). Because for each $p \in T_{\pi(x)}\mathrm{C} \cap B(\pi(x), R/2)$ there exists $q \in P(x)$ such that (5.124) is satisfied, we have that

$$\widetilde{\mathrm{d}}^{\pi(x),\frac{R}{2}}\left(T_{\pi(x)}\mathrm{C}, P(x)\right) \leq C|x|^{\beta_3}. \tag{5.125}$$

Because $T_{\pi(x)}\mathrm{C}$ is a cone through $\pi(x)$, Lemma A.1.2 and (5.125) give us

$$\mathrm{d}^{\pi(x),\frac{R}{2}}\left(T_{\pi(x)}\mathrm{C}, P(x)\right) \leq C|x|^{\beta_3}. \tag{5.126}$$

Lemma 5.3.7 follows from Lemma A.2.1.

□

Let $O = O(r_0, \delta)$ be the set

$$O = \left\{ x \in \mathbb{R}^n : |x_4| \leq \frac{r_0}{\sqrt{2}}, \big||\tau(x)| - |x_4|\big| \leq 2\delta|x_4| \right\}. \tag{5.127}$$

We note that $\mathrm{C} \cap B(0, r_0) = \mathrm{C} \cap O$. Lemma 5.3.6 allows us to prove that $\pi|_{\Sigma\cap O}$ is a lower Lipschitz map surjective onto $\mathrm{C} \cap O$ (see (5.93)).

Lemma 5.3.8. *Recall hypotheses (E0)-(E2). For r_0 small enough, $\pi|_{\Sigma\cap O}$ is lower Lipschitz and $\pi(\Sigma \cap O) = \mathrm{C} \cap O$.*

Proof. First we prove surjectivity which is an application of Theorem A.3.4. Fix $a \in B(0, r_0) \cap \mathrm{C}$, and consider the unit vector η_a. By (E0) and (E1), we have that if ε and σ are small enough, we may apply Theorem A.3.4. It follows that there exist a $|t| \leq \delta|a|$ and an $x \in \Sigma$ with $a + t\eta_a = x$. Next we claim that $x \in O$. It follows from the definition of x that $x_4 = a_4$ and $\tau(x) = \tau(a) + \tau(t\eta_a) = \tau(a) + t\eta_a$. Hence,

$$\big||\tau(x)| - |x_4|\big| = \big||\tau(a)| + |t\eta_a| - |a_4|\big| \leq \big|\tau(a) - |a_4|\big| + \delta|a| = 0 + \sqrt{2}\delta|a_4| = \sqrt{2}\delta|x_4|. \tag{5.128}$$

We finish by noting that $\pi(x) = a$ because $a \in \ell_x \cap \mathrm{C}$. Hence, $\pi(\mathrm{C} \cap O) = \Sigma \cap O$.

Next, we prove that $\pi|_{\Sigma\cap O}$ is lower Lipschitz for $x, y \in \Sigma \cap O$ sufficiently far apart. That is, we assume

$$|x - y| \geq \frac{\max(|x|, |y|)^{1+\gamma}}{A}. \tag{5.129}$$

By Lemma 5.3.3, we have that

$$|\pi(x) - x| \leq C|x|^{1+\beta_1}, \quad |\pi(y) - y| \leq C|y|^{1+\beta_1}. \tag{5.130}$$

By applying (5.129) and (5.130), we get

$$\begin{aligned}|\pi(x)-\pi(y)| &\geq |x-y|-|\pi(x)-x|-|\pi(y)-y| \geq |x-y| - C\left(|x|^{1+\beta_1}+|y|^{1+\beta_1}\right)\\ &\geq |x-y| - C\max(|x|,|y|)^{1+\beta_1} \geq |x-y| - Cr_0^{\beta_1-\gamma}\cdot\max(|x|,|y|)^{1+\gamma}\\ &\geq |x-y| - Cr_0^{\beta_1-\gamma}|x-y|\end{aligned} \tag{5.131}$$

(recall that $\beta_1 > \gamma$). Hence, for r_0 small enough, we get that

$$|\pi(x)-\pi(y)| \geq \frac{1}{2}|x-y| \quad \text{for } x,y\in\Sigma\cap O \text{ such that } |x-y| \geq \frac{\max(|x|,|y|)^{1+\gamma}}{A}. \tag{5.132}$$

To prove that $\pi|_{\Sigma\cap O}$ is lower Lipschitz on points which are very close together, we need two lemmas whose proofs appear in the appendix. The first one tells us how flatness of a set is perturbed by a C^2 diffeomorphism.

Lemma B.1.1. *Suppose that $U,V\subseteq\mathbb{R}^n$ are open sets, and $\psi\in C^2(U,V)$ is bijective and satisfies*

$$0<\lambda\leq\frac{|\psi(x)-\psi(y)|}{|x-y|}\leq\Lambda \qquad \text{for all } x,y\in U \tag{5.133}$$

and

$$||D^2\psi||_\infty = \sup_{x\in U}||D_x^2\psi|| < \infty. \tag{5.134}$$

Let $\Gamma\subseteq\mathbb{R}^n$, and $z\in\Gamma\cap U$, $B(z,r)\subseteq U$, P be a plane through z, and set $\widetilde{P}=D_z\psi(P-z)+\psi(z)$, $\widetilde{\Gamma}=\psi(\Gamma)$. Then

$$\mathrm{d}^{\psi(z),\lambda r}\left(\widetilde{\Gamma},\widetilde{P}\right) \leq \frac{||D^2\psi||_\infty}{2\lambda}r + \frac{\Lambda}{\lambda}\,\mathrm{d}^{z,r}\left(\Gamma,P\right). \tag{5.135}$$

Lemma B.1.2. *For $a\in\mathrm{C}\setminus\{0\}$, there exists a neighborhood $U\supseteq B(a,2|a|/A)$, $V\subseteq\mathbb{R}^3$ open, I an open interval with $0\in I$, and a smooth coordinate map $\psi^a: U\to V\times I$ such that $V\times\{0\}=\psi^a(\mathrm{C}\cap U)$ and $\widetilde{\pi}=\psi^a\circ\pi\circ(\psi^a)^{-1}$ is orthogonal projection onto $\mathbb{R}^3\times\{0\}$ (where π is the same map of Section 5; see (5.93)). Further, ψ^a satisifes the estimates*

$$\frac{1}{2}\leq\frac{|\psi^a(x)-\psi^a(y)|}{|x-y|}\leq 2 \qquad \text{for all } x,y\in U \tag{5.136}$$

and

$$||D^2\psi^a||_\infty = \sup_{x\in U}||D_x^2\psi^a|| \leq \frac{C}{|a|} \tag{5.137}$$

for some C independent of a.

We continue the proof of Lemma 5.3.8. Let $x \in \Sigma \cap O$. Let $a = \pi(x)$, let U, V, I, ψ^a, and $\widetilde{\pi}$ as in Lemma B.1.2. Set $R = |x|^{1+\gamma}/A$. Following the notation of Lemma B.1.1, let $\widetilde{\Sigma} = \psi^a(\Sigma \cap B(a, 2R))$. Following the notation of Lemma B.1.1, for $z \in \Sigma \cap B(a, 2R)$, $\widetilde{z} = \psi^a(z) \in \widetilde{\Sigma}$ and $0 < r \leq 2R$, we define $\widetilde{P}(\widetilde{z}, r) = \widetilde{P(z, 2r)} = D_z\psi^a(P(z, 2r) - z) + \widetilde{z}$. By Lemma B.1.1 and (5.136),

$$\mathrm{d}^{\widetilde{z},r}\left(\widetilde{\Sigma}, \widetilde{P}(\widetilde{z}, r)\right) \leq ||D^2\psi^a||_\infty \cdot 2r + 4\,\mathrm{d}^{z,2r}\left(\Sigma, P(z, 2r)\right). \tag{5.138}$$

for $0 < r \leq 2R$. By (5.137) and (E2), (5.138) gives $\mathrm{d}^{\widetilde{z},r}\left(\widetilde{\Sigma}, \widetilde{P}(\widetilde{z}, r)\right) \leq C|x|^{1+\gamma}/|a| + 4\delta$. Because $a = \pi(x)$, $|a| \geq |x| - |\pi(x) - x| \geq |x|/2$, and so

$$\mathrm{d}^{\widetilde{z},r}\left(\widetilde{\Sigma}, \widetilde{P}(\widetilde{z}, r)\right) \leq C|x|^\gamma + 4\delta \leq C|r_0|^\gamma + 4\delta. \tag{5.139}$$

Thus, we require that r_0 and δ be small enough that $C|r_0|^\gamma, 4\delta \leq 1/32$. So (5.139) gives us that

$$\mathrm{d}^{\widetilde{z},r}\left(\widetilde{\Sigma}, \widetilde{P}(\widetilde{z}, r)\right) \leq \frac{1}{16}. \tag{5.140}$$

Next we recall that Lemma B.1.2 tells us that $V \times \{0\} = \psi^a(\mathrm{C} \cap U)$. Similarly to before, we apply Lemmas 5.3.6 and B.1.1 to get that for $0 < r \leq 2R$,

$$\begin{aligned}\mathrm{d}^{\widetilde{z},R}\left(V \times \{0\}, \widetilde{P}(\widetilde{z}, r)\right) &\leq \frac{||D^2\psi^a||_\infty}{2 \cdot 1/2} + \frac{2}{1/2}\,\mathrm{d}^{z,r}\left(\mathrm{C}, P(z, r)\right) \\ &\leq C|x|^\gamma + C|x|^{\min(\beta_1 - \gamma, \beta_2(1+\gamma))} \leq C|x|^{\beta_3} \leq Cr_0^{\beta_3}.\end{aligned} \tag{5.141}$$

Note that e_4 is the normal vector to the plane $\mathbb{R}^3 \times \{0\}$, and let $\widetilde{\nu}_{\widetilde{z},r}$ be the normal vector (with positive 4th coordinate) to $\widetilde{P}(\widetilde{z}, r)$. Then Lemma A.2.1 and (5.141) guarantees that there is an r_0 small enough such that

$$|e_4 - \widetilde{\nu}_{\widetilde{z},r}| \leq \frac{1}{16}. \tag{5.142}$$

Let $y \in \Sigma \cap O$, $|x - y| \leq \frac{|x|^{1+\gamma}}{A}$. Let $\widetilde{x} = \psi^a(x)$ and $\widetilde{y} = \psi^a(y)$. In particular, we note that by (5.136), $\rho := |\widetilde{x} - \widetilde{y}| \leq 2|y - z| \leq 2R$, and so $\widetilde{P}(\widetilde{x}, \rho)$ is defined. Because $\widetilde{\pi}$ is orthogonal projection onto $\mathbb{R}^3 \times \{0\} = \langle e_4 \rangle^\perp$, we get

$$|\widetilde{x} - \widetilde{y}|^2 = |\widetilde{\pi}(\widetilde{x}) - \widetilde{\pi}(\widetilde{y})|^2 + |\langle \widetilde{x} - \widetilde{y}, e_4 \rangle|^2. \tag{5.143}$$

We compute by (5.140) and (5.142) that

$$|\langle \widetilde{x}-\widetilde{y}, e_4\rangle| \leq |\langle \widetilde{x}-\widetilde{y}, \widetilde{\nu}_{\widetilde{x},\rho}\rangle| + |\langle \widetilde{x}-\widetilde{y}, e_4-\widetilde{\nu}_{\widetilde{x},\rho}\rangle| \leq \frac{1}{16}\rho + \frac{1}{16}|\widetilde{x}-\widetilde{y}| = \frac{1}{8}|\widetilde{x}-\widetilde{y}|. \quad (5.144)$$

We apply (5.144) to (5.143) to get $|\widetilde{\pi}(\widetilde{x})-\widetilde{\pi}(\widetilde{y})|^2 = |\widetilde{x}-\widetilde{y}|^2 - |\langle \widetilde{x}-\widetilde{y}, e_4\rangle|^2 \geq \frac{63}{64}|\widetilde{x}-\widetilde{y}|^2 \geq \frac{63}{64\cdot 4}|x-y|^2$. Hence, $\pi|_{\Sigma\cap O}$ is lower Lipschitz.

□

Hence, we may define $\varphi : \mathrm{C}\cap O \to \Sigma\cap O$ by $\varphi = \pi^{-1}_{\Sigma\cap O}$. Because $\pi_{\Sigma\cap O}$ is lower Lipschitz, φ is upper Lipschitz. We use φ as our parametrization of Σ in a neighborhood of 0. We now begin the process of extending φ to a $C^{1,\beta}$ map on a neighborhood of 0. We will employ a modification of the Whitney Extension Theorem, which we state here explicitly for the reader's convenience. For $f \in C^k(\mathbb{R}^m \to \mathbb{R}^\ell)$, we let $D^k_a f$ be the k-linear map from $(\mathbb{R}^m)^k$ of partial derivatives at a, where by convention we take $D^0_a f = f(a)$.

Theorem 5.3.9. ($C^{k,\beta}$ Whitney Extension Theorem)
Let $\beta > 0$, $k, l, m \in \mathbb{N}$, $A \subseteq \mathbb{R}^m$ be closed, and for each $a \in A$ a polynomial $P_a : \mathbb{R}^m \to \mathbb{R}^l$ such that $\deg P_a \leq k$. *Define for $K \subseteq A$, $r > 0$, $0 \leq i \leq k$,*

$$\rho_i(K,r) = \sup\left\{\frac{||D^i_b P_b - D^i_b P_a||}{|a-b|^{k-i}} : a,b \in A, |a-b| \leq r\right\}. \quad (5.145)$$

If for each compact $K \subseteq A$ and each $0 \leq i \leq k$

$$\rho_i(K,r) \leq Cr^\beta \quad (5.146)$$

then there exists $\varphi \in C^{k,\beta}_{loc}(\mathbb{R}^m \to \mathbb{R}^l)$ such that for all $a \in A$ and $0 \leq i \leq k$, $D^i\varphi(a) = D^iP_a(a)$.

We first say a few words on this theorem. Although the extension in the theorem as stated is $C^{k,\beta}_{loc}(\mathbb{R}^m \to \mathbb{R}^\ell)$, we will only be interested in a $C^{1,\beta}$ extension of φ to a neighborhood of 0. The theorem is presented this way to be consistent with Federer.

We define the polynomials we will use for our analysis. We first set some notation. For $a \in \mathrm{C}\setminus\{0\}$, let $r_a = a/|a|$ be the unit radial tangent vector. Let ν_a be the inward pointing unit normal vector to C at a as defined in (5.97). A vector θ_a at a orthogonal to both r_a and ν_a will be said to be of type θ. The motivation is that a vector of type θ is tangent to the cross section C_{a_4}. For $x \in \Sigma \cap O \setminus \{0\}$ and $0 < r \leq |x|^{1+\gamma}/A$, let $L(x,r) = P(x,r) - x$ and $L(x) = P(x) - x$ be the approximating planes recentered to be vector spaces. For $a \in C \cap 0 \setminus \{0\}$, let λ_a be the unit normal vector to $L(\varphi(a))$. For $a, b \in C$, let ϕ_a be projection in the direction of η_a onto $L(\varphi(a))$, and $\phi_{a,b}$ be projection in the direction of η_a onto $L(\varphi(b))$. Note that $\phi_a = \phi_{a,a}$. Recall that $\tau(x) = (x_1, x_2, x_3)$.

We define M_a by

$$M_a(r_a) = \phi_a(r_a), \;\; M_a(\nu_a) = \nu_a, \;\; M_a(\theta_a) = \phi_a(\theta_a)\frac{|\tau(\varphi(a))|}{|\tau(a)|}\,, \tag{5.147}$$

where θ_a is any vector of type θ. We also set $R(a) = \frac{|\tau(\varphi(a))|}{|\tau(a)|}$, which yields the slightly cleaner expression $M_a(\theta_a) = \phi_a(\theta_a)R(a)$ for any vector θ_a of type θ. Define also $M_0 = \mathrm{Id}$. We then define, following the terminology of Theorem 5.3.9, a polynomial for each $a \in \Sigma \cap B(0, r_0)$, $P_a(x) = \varphi(a) + M_a(x - a)$. Note that $P_a(a) = \varphi(a)$.

Lemma 5.3.10. *Recall hypotheses (E0)-(E2).* $|\varphi(a) - a| = \sec(\pi/4)\ \mathrm{dist}(\varphi(a), C) \leq |a|^{1+\beta_1}$.

Proof. Follows immediately from Lemmas 5.3.2 and 5.3.3. □

Lemma 5.3.11. *For* $a, b \in C \cap O \setminus \{0\}$ *such that* $|a - b| \leq \frac{|a|^{1+\gamma}}{A}$,

(1) a vector v_a *is of type* θ *if and only if* $(v_a)_4 = 0$

(2) $|r_a - r_b| \leq C|a - b|^{\frac{\gamma}{1+\gamma}}$

(3) $|\nu_a - \nu_b| \leq C|a - b|^{\frac{\gamma}{1+\gamma}}$

(4) $|\eta_a - \eta_b| \leq C|a - b|^{\frac{\gamma}{1+\gamma}}$

(5) If θ_a *is a unity vector of type* θ *at* a*, then there exists a vector* θ_b *of type* θ *at* b *such that* $|\theta_a - \theta_b| \leq C|a - b|^{\frac{\gamma}{1+\gamma}}$.

Proof. We first prove (1). Let $u_a \in T_aC$. Then $u_a \perp \nu_a$. Note that $r_a + \nu_a = 2(a_4)/|a| \neq 0$. We then have that u_a is of type $\theta \Leftrightarrow u_a \cdot r_a = 0 \Leftrightarrow u_a \cdot (r_a + \nu_a) = 0 \Leftrightarrow (u_a)_4 = 0$.

We now prove (2)-(4). Recall that $r_a = a/|a|$. We observe the identity that for $w, x, y, z \in \mathbb{R}$,

$$xy - zw = \frac{1}{2}\left((x-z)(y+w) + (y-w)(x+z)\right). \tag{5.148}$$

Hence, we get that

$$\left|\frac{a}{|a|} - \frac{b}{|b|}\right| = \left|\frac{a|b| - |a|b}{|a||b|}\right| \leq \frac{1}{2}\left(\frac{|a-b|(|a|+|b|) - \big||a|-|b|\big| \cdot |a+b|}{|a||b|}\right) \leq \frac{|a-b|(|a|+|b|)}{2|a||b|}. \tag{5.149}$$

Recall that $|a-b| \leq |a|^{1+\gamma}/A$. This gives that $|b| \leq |b-a| + |a| \leq |a|(1+A)/A$, and similarly $|a| \leq |b|(1+A)/A$. Hence, $|b|$ and $|a|$ are within a constant factor of each other and so (5.149) gives that

$$\left|\frac{a}{|a|} - \frac{b}{|b|}\right| \leq C\frac{|a-b|}{|a|}. \tag{5.150}$$

Applying that $|a-b| \leq |a|^{1+\gamma}/A$, we get that $1/|a| \leq 1/(A|a-b|^{\frac{1}{1+\gamma}})$. Hence, we get from (5.150) that

$$\left|\frac{a}{|a|} - \frac{b}{|b|}\right| \leq C\frac{|a-b|}{|a-b|^{\frac{1}{1+\gamma}}} = C|a-b|^{\frac{\gamma}{1+\gamma}}. \tag{5.151}$$

From (5.151), claims (2), (3), and (4) follow.

We prove (5). Let θ_a be a unit vector of type θ in $T_a\mathrm{C} - a$. Let π_b be projection onto $T_b\mathrm{C} - b$ in the η_b direction. Define $\theta_b = \pi_b(\theta_a)$. By (1), $(\theta_a)_4 = 0$. Because π_b is projection onto $T_b\mathrm{C} - b$ in the η_b direction, we get that $(\theta_b)_4 = 0$ and $\theta_b \in T_b\mathrm{C} - b$. So by (1), θ_b is a vector of type θ at b. Further, we compute that

$$|\theta_a - \theta_b| = \sec(\measuredangle(\nu_b, \eta_b))\,\mathrm{dist}(\theta_a, T_b\mathrm{C} - b) \leq C\measuredangle(T_a\mathrm{C}, T_b\mathrm{C}). \tag{5.152}$$

By (2) and (A.17), we see that $|\theta_a - \theta_b| \leq C|a-b|^{\frac{\gamma}{1+\gamma}}$.

□

Lemma 5.3.12. *Recall hypotheses (E0)-(E2). For $a, b \in \mathrm{C}$ with $|a-b| \leq |a|^{1+\gamma}/A$,*

(1) $||\phi_{a,a} - \phi_{b,a}|| \leq C|a-b|^{\frac{\gamma}{1+\gamma}}$

(2) $||\phi_{a,a} - \phi_{a,b}|| \leq C|a-b|^{\beta_2}$

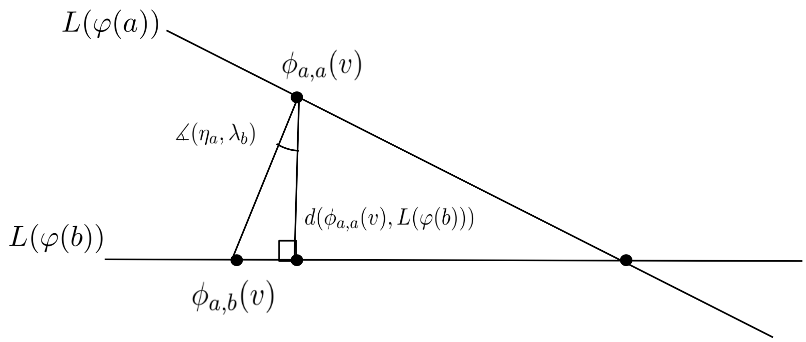

Figure 5.1: Diagram of (5.153)

(3) $||\phi_a - \phi_b|| \leq C|a-b|^{\min(\beta_2, \frac{\gamma}{1+\gamma})}$.

(4) $|R(a) - R(b)| \leq C|a-b|^{\frac{\gamma}{1+\gamma}}$.

Proof. First, we note that (1) follows from Lemma 5.3.11(2) and the definition of $\phi_{c,d}$.

We now prove (2). Let $v \in \mathbb{R}^4$, $|v| = 1$. We note because each of $\phi_{a,a}$ and $\phi_{a,b}$ is a projection in the η_a direction, $\phi_{a,a}(v) - \phi_{a,b}(v)$ is a scalar multiple of η_a. Thus, we have that

$$||\phi_{a,a}(v) - \phi_{a,b}(v)|| = \sec(\measuredangle(\eta_a, \lambda_b))\, \mathrm{dist}(\phi_{a,a}(v), L(\varphi(b))), \tag{5.153}$$

where we recall that λ_b is the normal vector to $L(\varphi(b))$ (see Figure 5.1).

Next, we note that

$$\measuredangle(\eta_a, \lambda_b) \leq \measuredangle(\eta_a, \eta_b) + \measuredangle(\eta_b, \nu_b) + \measuredangle(\nu_b, \lambda_b) \leq \frac{\pi}{3} \tag{5.154}$$

for r_0 small enough by Lemma 5.3.11(2), $\measuredangle(\eta_b, \nu_b) = \pi/4$, and Lemma 5.3.7. Thus, $\sec(\measuredangle(\eta_a, \lambda_b)) \leq 2$. In addition, for r_0 small enough we have that $\measuredangle(\nu_a, \lambda_a) \leq 5\pi/12$ (see (5.154) and Lemma 5.3.7), so $|\phi_{a,a}(v)| \leq R$ for some R (independent of a and b). Hence, (5.153) and (5.154) gives us that

$$||\phi_{a,a}(v) - \phi_{a,b}(v)|| \leq C\, \mathrm{D}^{0,R}\left[L(\varphi(a)), L(\varphi(b))\right] \leq C\, \mathrm{D}^{0,1}\left[L(\varphi(a)), L(\varphi(b))\right]. \tag{5.155}$$

Because φ is Lipschitz, (A.17), Corollary 5.3.5 and (5.155) tell us that

$$||\phi_{a,a}(v) - \phi_{a,b}(v)|| \leq C|\varphi(a) - \varphi(b)|^{\beta_2} \leq C|a-b|^{\beta_2}. \tag{5.156}$$

To prove (3), we apply (1) and (2) to get

$$||\phi_a - \phi_b|| \leq ||\phi_{a,a} - \phi_{a,b}|| + ||\phi_{a,b} - \phi_{b,b}|| \leq C|a-b|^{\min(\beta_2, \frac{\gamma}{1+\gamma})}. \tag{5.157}$$

We now prove (4). Applying the definition of R, we get

$$|R(a) - R(b)| = \left| \frac{|\tau(\varphi(a))|}{|\tau(a)|} - \frac{|\tau(\varphi(b))|}{|\tau(b)|} \right| = \left| \frac{|\tau(\varphi(a))|\,|\tau(b)| - |\tau(\varphi(b))|\,|\tau(a)|}{|\tau(a)||\tau(b)|} \right|. \tag{5.158}$$

We reobserve the identity that for $w, x, y, z \in \mathbb{R}$,

$$xy - zw = \frac{1}{2}\left((x-z)(y+w) + (y-w)(x+z)\right). \tag{5.159}$$

Applying (5.159) to (5.158) gives

$$\begin{aligned} |R(a) - R(b)| \leq \frac{1}{2|\tau(a)||\tau(b)|} \Big(&\big||\tau(\varphi(a))| - |\tau(\varphi(b))|\big|\big(|\tau(a)| + |\tau(b)|\big) \\ &+ \big||\tau(a)| - |\tau(b)|\big|\big(|\tau(\varphi(a))| + |\tau(\varphi(b))|\big)\Big). \end{aligned} \tag{5.160}$$

We note that because $|\tau(a)| = |a|/\sqrt{2}$ because $a \in \mathrm{C}$, and the same holds for b. We then apply that φ is Lipschitz (as well as $\varphi(0) = 0$), and the fact that $|\tau(x) - \tau(y)| \leq |x - y|$ to (5.160) get that

$$|R(a) - R(b)| \leq \frac{C}{|a||b|}\Big(|a-b|(|a| + |b|)\Big) \tag{5.161}$$

Recall that we showed that $|a|$ and $|b|$ are within a constant factor of eachother (while proving (5.150), so (5.161) gives

$$|R(a) - R(b)| \leq C\frac{|a-b|}{|a|} \leq C\frac{|a-b|}{|a-b|^{\frac{1}{1+\gamma}}} = C|a-b|^{\frac{\gamma}{1+\gamma}}. \tag{5.162}$$

□

We note that the argument used in (5.152) and (5.153) will recur in many variations in the proofs to come. The reader who glazed over this point is encouraged to review it.

Lemma 5.3.13. *Recall hypotheses (E0)-(E2). For $a \in C \cap O$, $||M_a - M_0|| \leq C|a|^{\beta_3}$.*

Proof. Note that at a we can choose an orthonormal basis $\mathcal{B}_a = \{r_a, \nu_a, \theta_a^1, \theta_a^2\}$ where r_a and ν_a are the radial and normal vectors as before, and θ_a^i is a vector of type θ. We thus check that each vector $v \in \mathcal{B}_a$ will satisfy the bound

$$|M_a(v) - M_0(v)| \leq C|a|^{\beta_3}, \tag{5.163}$$

from which the result follows. Recall that by definition $M_0 = \mathrm{Id}$.

First, we consider the easiest case, ν_a. By definition $M_a(\nu_a) = \nu_a$, and so $M_a(\nu_a) - \nu_a = 0$.

Next, consider $M_a(r_a) - r_a$. By definition, $M_a(r_a) = \phi_a(r_a)$. Because ϕ_a is a projection in the η_a direction, we get that

$$|\phi_a(r_a) - r_a| = \sec(\measuredangle(\eta_a, \lambda_a)) \operatorname{dist}(r_a, L(\varphi(a))) \leq C\, \mathrm{D}^{0,1}\left[T_a\mathrm{C} - a, L(\varphi(a))\right] \leq C|a|^{\beta_3}. \tag{5.164}$$

Finally, we consider θ_a a vector of type θ. Recall that $M_a(\theta_a) = \frac{|\tau(\varphi(a))|}{|\tau(a)|}\phi_a(\theta_a)$. We compute

$$|M_a(\theta_a) - \theta_a| = \left|\frac{|\tau(\varphi(a))|}{|\tau(a)|}\phi_a(\theta_a) - \theta_a\right| \leq \left|\frac{|\tau(\varphi(a))|}{|\tau(a)|} - 1\right| |\phi_a(\theta_a)| + |\phi_a(\theta_a) - \theta_a|. \tag{5.165}$$

From Lemma 5.3.10 we get that

$$\left|\frac{|\tau(\varphi(a))|}{|\tau(a)|} - 1\right| = \frac{\big||\tau(\varphi(a))| - |\tau(a)|\big|}{|\tau(a)|} \leq C\frac{|\varphi(a) - a|}{|a|} \leq C|a|^{\beta_1}. \tag{5.166}$$

Applying (5.166) to (5.165), we get

$$\begin{aligned} |M_a(\theta_a) - \theta_a| &\leq C|a|^{\beta_1} + \sec(\measuredangle(\eta_a, \lambda_a)) \operatorname{dist}(\theta_a, L(\varphi(a))) \\ &\leq C|a|^{\beta_1} + C\, \mathrm{D}^{0,1}\left[T_a\mathrm{C} - a, L(\varphi(a))\right] \leq C|a|^{\beta_3}. \end{aligned} \tag{5.167}$$

□

Lemma 5.3.14. *Recall hypotheses (E0)-(E2). For $a, b \in \mathrm{C} \cap O$, we have that $||M_a - M_b|| \leq C|a - b|^{\frac{\beta_3}{1+\gamma}}$. Or in the language of Theorem 5.3.9, $\rho_1(\mathrm{C} \cap O, r) \leq Cr^{\frac{\beta_3}{1+\gamma}}$ (recall (5.145), and note that $\mathrm{C} \cap O$ is compact).*

Proof. Recall that $\beta_3 = \min(\beta_1 - \gamma, \beta_2(1+\gamma), \gamma)$. We break the proof into two scales. First, we assume that $|a - b| \geq \max(|a|, |b|)^{1+\gamma}/A$. Then Lemma 5.3.13 tells us that

$$||M_a - M_b|| \leq ||M_a - M_0|| + ||M_b - M_0|| \leq C\left(|a|^{\beta_3} + |b|^{\beta_3}\right) \leq C|a - b|^{\frac{\beta_3}{1+\gamma}}. \tag{5.168}$$

Let us now assume that $|a - b| \leq |a|^{1+\gamma}/A$. As in Lemma 5.3.13, we will consider the radial, normal, and type θ vectors separately. By Lemma 5.3.11(3), we have that

$|\nu_a - \nu_b| \le C|a-b|^{\frac{\gamma}{1+\gamma}}$. We recall that $M_a(\nu_a) = \nu_a$. Thus, we compute

$$\begin{aligned} |M_a(\nu_a) - M_b(\nu_a)| &\le |M_a(\nu_a) - M_b(\nu_b)| + |M_b(\nu_b - \nu_a)| \le |\nu_a - \nu_b| + ||M_b|| \cdot |\nu_a - \nu_b| \\ &\le C|a-b|^{\frac{\gamma}{1+\gamma}} \end{aligned} \tag{5.169}$$

(note $||M_b|| \le ||M_b - M_0|| + 1 \le C$ by Lemma 5.3.13).

We now show that

$$|M_a(r_a) - M_b(r_a)| \le C|a-b|^{\frac{\beta_3}{1+\gamma}}. \tag{5.170}$$

By Lemma 5.3.11(2), we have that $|r_a - r_b| \le C|a-b|^{\frac{\gamma}{1+\gamma}}$. We thus compute

$$\begin{aligned} |M_a(r_a) - M_b(r_a)| &\le |M_a(r_a) - M_b(r_b)| + |M_b(r_b - r_a)| \le |\phi_a(r_a) - \phi_b(r_b)| + ||M_b|| |r_b - r_a| \\ &\le |\phi_a(r_a) - \phi_b(r_b)| + C|b-a|^{\frac{\gamma}{1+\gamma}}. \end{aligned} \tag{5.171}$$

We now consider $|\phi_a(r_a) - \phi_b(r_b)|$. Applying By Lemma 5.3.11(2) and Lemma 5.3.12, we get

$$\begin{aligned} |\phi_a(r_a) - \phi_b(r_b)| &\le |\phi_a(r_a - r_b)| + |(\phi_a - \phi_b)(r_b)| \le ||\phi_a|| \cdot |r_a - r_b| + ||\phi_a - \phi_b|| \cdot |r_b| \\ &\le C|a-b|^{\min(\beta_2, \frac{\gamma}{1+\gamma})}. \end{aligned} \tag{5.172}$$

Coupling (5.171) and (5.172), we prove (5.170).

Finally, we consider vectors of type θ. Let θ_a be a vector of type θ at a. By Lemma 5.3.11(5) there is a vector θ_b of type θ at b such that $|\theta_a - \theta_b| \le C|a-b|^{\frac{\gamma}{1+\gamma}}$. We now show that

$$|M_a(\theta_a) - M_b(\theta_a)| \le C|a-b|^{\frac{\beta_3}{1+\gamma}}. \tag{5.173}$$

We compute that

$$\begin{aligned} |M_a(\theta_a) - M_b(\theta_a)| &\le |M_a(\theta_a) - M_b(\theta_b)| + |M_b(\theta_b - \theta_a)| \le |M_a(\theta_a) - M_b(\theta_b)| + ||M_b|| \cdot |\theta_b - \theta_a| \\ &\le |R(a)\phi_a(\theta_a) - R(b)\phi_b(\theta_b)| + C|b-a|^{\frac{\gamma}{1+\gamma}}. \end{aligned} \tag{5.174}$$

We now consider $|R(a)\phi_a(\theta_a) - R(b)\phi_b(\theta_b)|$. We reobserve the identity

$$xy - zw = \frac{1}{2}\left((x-z)(y+w) + (y-w)(x+z)\right). \tag{5.175}$$

Applying (5.175) to $|R(a)\phi_a(\theta_a) - R(b)\phi_b(\theta_b)|$ gives that

$$|R(a)\phi_a(\theta_a) - R(b)\phi_b(\theta_b)| \leq |R(a) - R(b)|(|\phi_a(\theta_a)| + |\phi_b(\theta_b)|) + |\phi_a(\theta_a) - \phi_b(\theta_b)|(R(a) + R(b)). \tag{5.176}$$

We then apply that $|R(a)|$, $|R(b)|$, $||\phi_a||$ and $||\phi_b||$ are all bounded, plus Lemma 5.3.12(4) to (5.176) to get

$$|R(a)\phi_a(\theta_a) - R(b)\phi_b(\theta_b)| \leq C|\phi_a(\theta_a) - \phi_b(\theta_b)| + C|a-b|^{\frac{\gamma}{1+\gamma}}. \tag{5.177}$$

Thus to establish (5.173) (and finish the proof), we establish

$$|\phi_a(\theta_a) - \phi_b(\theta_b)| \leq C|a-b|^{\frac{\beta_3}{1+\gamma}}. \tag{5.178}$$

Applying $|\theta_a - \theta_b| \leq C|a-b|^{\frac{\gamma}{1+\gamma}}$ and Lemma 5.3.12, we compute that

$$\begin{aligned} |\phi_a(\theta_a) - \phi_b(\theta_b)| &\leq |\phi_a(\theta_a - \theta_b)| + |(\phi_a - \phi_b)(\theta_b)| \leq ||\phi_a|| \cdot |\theta_a - \theta_b| + ||\phi_a - \phi_b|| \cdot |\theta_b| \\ &\leq C|a-b|^{\min(\beta_2, \frac{\gamma}{1+\gamma})} \end{aligned} \tag{5.179}$$

and conclude that (5.173) holds. □

Lemma 5.3.15. *Recall hypotheses (E0)-(E2). For $a, b \in \mathrm{C} \cap O$, we have*

$$\frac{|P_b(b) - P_a(b)|}{|b-a|} \leq C|b-a|^{\frac{\beta_3}{1+\gamma}}. \tag{5.180}$$

Or, in the notation of Theorem 5.3.9, we have that $\rho_0(\mathrm{C} \cap O, r) \leq Cr^{\frac{\beta_3}{1+\gamma}}$ (recall (5.145), and note that $\mathrm{C} \cap O$ is compact).

Proof. Applying the definition of the polynomials P_a and P_b, we get

$$P_b(b) - P_a(b) = \varphi(b) + M_b(b-b) - \varphi(a) - M_a(b-a) = \varphi(b) - \varphi(a) - M_a(b-a). \tag{5.181}$$

To begin, we consider the case where $|a-b| \geq \max(|a|, |b|)^{1+\gamma}/A$. We compute that

$$|\varphi(b) - \varphi(a) - M_a(b-a)| \leq |b-a-M_a(b-a)| + |\varphi(b) - b| + |\varphi(a) - a|. \tag{5.182}$$

By Lemmas 5.3.10 and 5.3.13, we get that

$$\begin{aligned}|b-a-M_a(b-a)|+|\varphi(b)-b|+|\varphi(a)-a| &\le ||M_a-\mathrm{Id}||\cdot|b-a|+C|b|^{1+\beta_1}+C|a|^{1+\beta_1}\\ &\le |a|^{\beta_3}|b-a|+C|a-b|^{\frac{1+\beta_1}{1+\gamma}}\le C|b-a|^{\frac{\beta_3}{1+\gamma}}.\end{aligned} \tag{5.183}$$

Assume now that $a, b \in \mathrm{C} \cap O$ and $|a-b| \le |a|^{1+\gamma}/A$. First, we define some ways that a and b may differ from each other. Let π_a be projection in the η_a direction onto $T_a\mathrm{C} - a$.

(i) We say that a and b are *radially separated* if $b-a$ is a radial vector.

(ii) We say that a and b are θ *separated* if $\pi_a(b-a)$ is a vector of type θ.

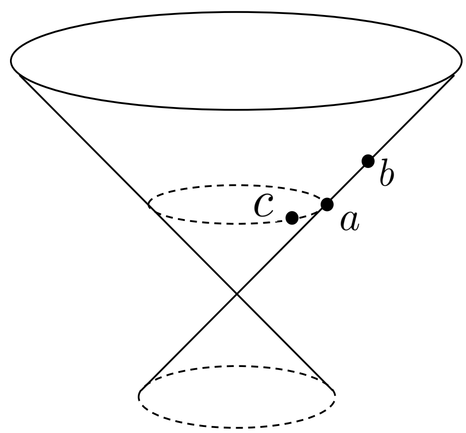

Figure 5.2: The points a and b are radially separated, a and c are θ separated

Note that if $b-a = cr_a$, then $\pi_a(b-a) = b-a$, and so conditions (i) and (ii) have more symmetry than may initially appear. We develop an alternative characterization of condition (ii), that

$$a \text{ and } b \text{ are } \theta \text{ separated} \quad \Leftrightarrow \quad a_4 = b_4. \tag{5.184}$$

To see this, we first note that for all vectors $v \in T_a\mathrm{C} - a$, that $v \cdot \nu_a = 0$. Also, $\nu_a + r_a = 2a_4/|a| \neq 0$ because $a \in \mathrm{C} \setminus \{0\}$. Also, we note that because π_a is projection in the η_a

direction and $(\eta_a)_4 = 0$, π_a does not change the 4th coordinate. Thus, we have that

$$\begin{aligned} a_4 = b_4 &\Leftrightarrow \pi_a(b-a)_4 = 0 \Leftrightarrow \pi_a(b-a)\cdot(\nu_a + r_a) = 0 \Leftrightarrow \pi_a(b-a)\cdot r_a = 0 \\ &\Leftrightarrow a \text{ and } b \text{ are } \theta \text{ separated,} \end{aligned} \tag{5.185}$$

establishing (5.184).

First, assume that a and b are radially separated. Then because $b-a$ is a mulitple of r_a, we apply the definition of M_a to get that

$$P_b(b) - P_a(b) = \varphi(b) - \varphi(a) - M_a(b-a) = \varphi(b) - \varphi(a) - \phi_a(b-a). \tag{5.186}$$

Note that when a and b are radially separated, $a/|a| = b/|b|$. So $r_a = r_b$ and $\eta_a = \eta_b$. We now use this to claim that

$$\varphi(a) + \phi_a(b-a) \in \ell_b. \tag{5.187}$$

To see this, we expand

$$\varphi(a) + \phi_a(b-a) = \big(\varphi(a) - a\big) + \big(\phi_a(b-a) - (b-a)\big) + b. \tag{5.188}$$

Because φ is the inverse of π which was projection in the η_a direction, $\varphi(a) - a$ is a multiple of η_a, which in this case satisfies $\eta_a = \eta_b$. Because ϕ_a is a projection in the η_a direction, we have that $\phi_a(b-a) - (b-a)$ is a scalar multiple of η_a, and $\eta_a = \eta_b$. Thus, from (5.188), we have that for some $s \in \mathbb{R}$,

$$\varphi(a) + \phi_a(b-a) = b + s\eta_b \in \ell_b \tag{5.189}$$

(recall (5.96)). Because $\varphi(b) \in \ell_b$, we have that $\varphi(b) - \varphi(a) - \phi_a(b-a)$ is a scalar multiple of η_b. Recall that λ_a is the normal vector to $P(\varphi(a))$. Thus, because $\varphi(a) + \phi_a(b-a) \in P(\varphi(a))$ (because ϕ_a is projection into $L(\varphi(a))$), we have that

$$|\varphi(b) - \varphi(a) - \phi_a(b-a)| = \sec(\measuredangle(\eta_b, \lambda_a)) \operatorname{dist}(\varphi(b), P(\varphi(a)) \leq C|b-a|^{1+\beta_2} \tag{5.190}$$

by Corollary 5.3.5 and φ being Lipschitz (Lemma 5.3.8). Thus, we have established (a stronger inequality than) (5.180) for a and b radially separated.

Next, we assume that a and b are θ separated. In this case, many of the simplifications that we were able to make in the radial case will not hold true, but will be true up to

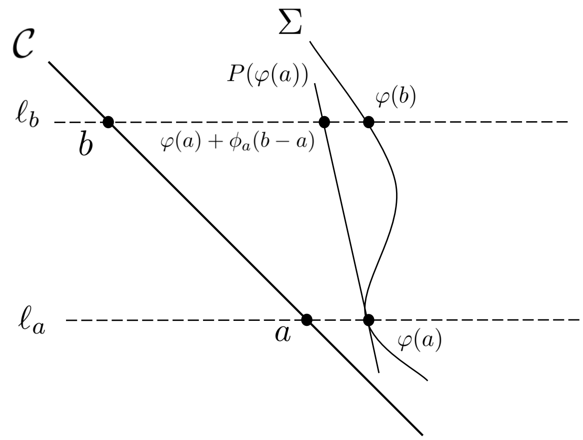

Figure 5.3: The argument for radially separated points.

$O\big(|b-a|^{1+\frac{\beta_3}{1+\gamma}}\big)$. Thus, while the core ideas remain the same, several more estimates must be applied. We begin by applying the assumption that a and b are θ separated to get

$$\begin{aligned} |\varphi(b) - \varphi(a) - M_a(b-a)| &\leq |\varphi(b) - \varphi(a) - M_a(\pi_a(b-a))| + |M_a(\pi_a(b-a)) - M_a(b-a)| \\ &\leq |\varphi(b) - \varphi(a) - R(a)\phi_a(\pi_a(b-a))| + ||M_a|| \cdot |\pi_a(b-a) - (b-a)|. \end{aligned} \tag{5.191}$$

By Lemma 4.3.1(2) and that π_a is projection onto $T_a\mathrm{C}$ in the η_a direction, we have that

$$|\pi_a(b-a) - (b-a)| \leq \sec(\measuredangle(\eta_a, \nu_a)) \operatorname{dist}(b, T_a\mathrm{C}) \leq C\frac{|b-a|^2}{|a|} \leq C|b-a|^{1+\frac{\gamma}{1+\gamma}}. \tag{5.192}$$

Thus applying (5.192) to (5.191), we get that

$$|\varphi(b) - \varphi(a) - M_a(b-a)| \leq |\varphi(b) - \varphi(a) - R(a)\phi_a(\pi_a(b-a))| + C|b-a|^{1+\frac{\gamma}{1+\gamma}}. \tag{5.193}$$

Thus, we strive to establish

$$|\varphi(b) - \varphi(a) - R(a)\phi_a(\pi_a(b-a))| \leq C|b-a|^{1+\frac{\beta_3}{1+\gamma}}, \tag{5.194}$$

which by (5.193) will establish (5.180) for a and b which are θ separated.

To begin proving (5.194), we note that because both ϕ_a and π_a are projections in the η_a direction, that $\phi_a(\pi_a(b-a)) = \phi_a(b-a)$. We use this and the linearity of ϕ_a to get

$$\varphi(b) - \varphi(a) - R(a)\phi_a(\pi_a(b-a)) = \varphi(b) - \varphi(a) - \phi_a(R(a)(b-a)). \tag{5.195}$$

Next, we apply Lemma 5.3.12(1) to get

$$\begin{aligned} |\varphi(b) - \varphi(a) - \phi_a(R(a)(b-a))| &\leq |\varphi(b) - \varphi(a) - \phi_{b,a}(R(a)(b-a))| + |(\phi_{b,a} - \phi_a)(R(a)(b-a))| \\ &\leq |\varphi(b) - \varphi(a) - \phi_{b,a}(R(a)(b-a))| + C|b-a|^{1+\frac{\gamma}{1+\gamma}}. \end{aligned} \tag{5.196}$$

Similarly to the radial case, we now claim that

$$\varphi(a) + \phi_{b,a}(R(a)(b-a)) \in \ell_b. \tag{5.197}$$

Recall that by (5.184), $a_4 = b_4$. Note that $a + |\tau(a)|\eta_a = (0, a_4)$, where 0 here represents $0 \in \mathbb{R}^3$. Thus, we have that $\ell_a \ni (0, a_4) = (0, b_4) \in \ell_b$. For $x, y, z \in \mathbb{R}^4$ not colinear, let $\Delta x, y, z$ be the triangle with corners x, y, and z. Consider $\Delta(0, a_4), a, b$. Let z be the point so that $\Delta(0, a_4), \varphi(a), z$ is similar to $\Delta(0, a_4), a, b$ (see Fig. 2). Then we have that $z \in \ell_b$. Further, because the length of side $(0, a_4), a$ is $|\tau(a)|$ and the length of side $(0, a_4), \varphi(a)$ is $|\tau(\varphi(a))|$, we get that $z = \varphi(a) + \frac{|\tau(\varphi(a))|}{|\tau(a)|}(b-a) = \varphi(a) + R(a)(b-a)$. Thus, $\varphi(a) + R(a)(b-a) \in \ell_b$. Further, since $\phi_{b,a}$ is a projection in the η_b direction, we have that (5.197) holds.

Note $\varphi(b) \in \ell_b$. By (5.197), we have that $\varphi(b) - \varphi(a) - \phi_{b,a}(R(a)(b-a))$ is a scalar multiple of η_b and $\varphi(a) + \phi_{b,a}(R(a)(b-a)) \in P(\varphi(a))$. Thus, as in the radial case, we get that

$$|\varphi(b) - \varphi(a) - \phi_{b,a}(R(a)(b-a))| = \sec(\measuredangle(\eta_b, \lambda_a)) \operatorname{dist}(\varphi(b), P(\varphi(a))) \leq C|b-a|^{1+\beta_2}. \tag{5.198}$$

Thus, we have established (5.194), and thus established (a slightly stronger version of) (5.180) for θ separated points.

Finally, we consider general points $a, b \in \mathrm{C} \cap O$ with $|a-b| \leq |a|^{1+\gamma}/A$. We define $c = (\tau(a), b_4)$. Note that a and c are radially separated, b and c are θ separated, and

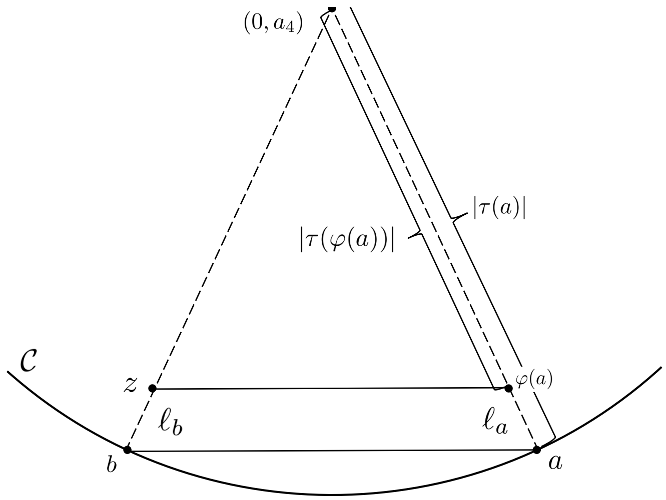

Figure 5.4: The point z in the argument above; $z = \varphi(a) + \frac{|\tau(\varphi(a))|}{|\tau(a)|}(b - a)$

$|a - c|, |c - b| \leq |a - b|$. We compute $P_b(b) - P_a(b) = P_b(b) - P_c(b) + P_c(b) - P_a(b)$. We expand to get

$$\begin{aligned} P_c(b) - P_a(b) &= \varphi(c) + M_c(b - c) - \varphi(a) - M_a(b - a) \\ &= \varphi(c) - \varphi(a) - M_a(c - a) + M_a(c - a) - M_a(b - a) + M_c(b - c) \\ &= P_c(c) - P_a(c) + M_a(c - b) - M_c(c - b). \end{aligned} \tag{5.199}$$

Thus, by (5.199), we get that

$$|P_b(b) - P_a(b)| \leq |P_b(b) - P_c(b)| + |P_c(c) - P_a(c)| + ||M_a - M_c|| \cdot |c - b|. \tag{5.200}$$

We now use that (5.180) holds for radially and θ separated points, Lemma 5.3.14, and (5.200) get that

$$|P_b(b) - P_a(b)| \leq C|b - c|^{1+\frac{\beta_3}{1+\gamma}} + C|c - a|^{1+\frac{\beta_3}{1+\gamma}} + C|a - c|^{\frac{\beta_3}{1+\gamma}}|c - b|. \tag{5.201}$$

We now apply that $|b - c|, |c - a| \leq |a - b|$ to get that $|P_b(b) - P_a(b)| \leq C|a - b|^{1+\frac{\beta_3}{1+\gamma}}$ for all points $a, b \in \mathrm{C} \cap O$.

□

We now have gathered all of the estimates necessary to prove Theorem 5.3.1.

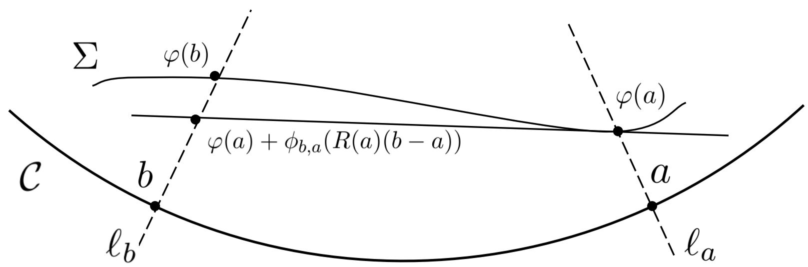

Figure 5.5: The argument for θ separated points.

Proof of Theorem 5.3.1. Let $\beta = \beta_3/(1+\gamma)$. First, note that $\varphi(a) = P_a(a)$. By Lemmas 5.3.14 and 5.3.15, we have that Theorem 5.3.9 says that φ extends to a $C^{1,\beta}$ map on $\mathbb{R}^4$ (which we also call φ) such that $\varphi(a) = P_a(a)$ and $D_a\varphi = D_aP_a$ for all $a \in \mathrm{C} \cap O$. Because φ is a bijection from $\mathrm{C} \cap O$ to $\Sigma \cap O$ (see Lemma 5.3.8), and $D_aP_a = M_a$ which is always of full rank (see definition of M_a), we have that there is some open set U containing $\mathrm{C} \cap O$ such that φ is a diffeomorphism on U. Taking $U' = \varphi(U)$ completes the proof.

□

Remark 5.3.16. *We finish by reminding ourselves that Theorem 5.2.1 is a local theorem. For example, consider a set shaped like a candy wrapper, as shown in Figure 5.6 (which is of course a dimension short) shows a set which at every point is smoothly parametrized by a KP cone or a plane.*

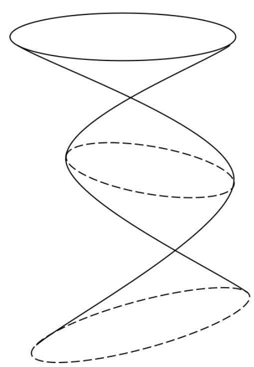

Figure 5.6: A "candy-wrapper" set which is locally smoothly parametrized by planes and KP cones.

Appendix A

SOME FACTS ABOUT SETS APPROXIMATED BY PLANES AND KP CONES

In this appendix,we record some properties of sets which are well approximated by planes and KP cones. In Section A.1, we show some nice properties of the Walkup-Wets distance and relative Hausdorff distance. In Section A.2, we record some lemmas about the geometry of planes. In Section A.3, we study sets well approximated by KP cones with an eye for the interactions between $\Theta_{\Sigma}^{\mathcal{C}_0}$, $\Theta_{\Sigma}^{\mathcal{C}}$, and $\Theta_{\Sigma}^{\mathcal{G}}$ at different points and scales.

A.1 Walkup-Wets Distance and Hausdorff Distance for Dilation Invariant Sets

Our goal in this section is to study the geometry of sets which are well approximated by cones; that is, dilation invariant sets.

Definition A.1.1. A set $C \subseteq \mathbb{R}^n$ is a *cone based at* y if $y \in C$ and $C - y$ satisfies that $s(C-y) = C - y$ for all $s > 0$, .

Example. Recall that a KP cone based at y is a set C which in some orthonormal coordinates (x_i) centered at the origin satisfies $\mathrm{C} - y = \{x_4^2 = x_1^2 + x_2^2 + x_3^2\}$. Note that a KP cone based at y is a cone based at y and that a plane including y is a cone based at y.

Lemma A.1.2. *Let C be a cone based at $y \in \mathbb{R}^n$ and $\Sigma \subseteq \mathbb{R}^n$ be any set. Then for any $r > 0$ such that $B(y,r) \cap \Sigma \neq \emptyset$, we have that*

$$\mathrm{d}^{y,r}(\Sigma, C) = \widetilde{\mathrm{d}}^{y,r}(\Sigma, C) \quad \textit{and} \quad \widetilde{\mathrm{d}}^{y,r}(C,\Sigma) \leq \mathrm{d}^{y,r}(C,\Sigma) \leq 2\,\widetilde{\mathrm{d}}^{y,r}(C,\Sigma)\,. \tag{A.1}$$

In particular,

$$\widetilde{\mathrm{D}}^{y,r}[C,\Sigma] \leq \mathrm{D}^{y,r}[C,\Sigma] \leq 2\,\widetilde{\mathrm{D}}^{y,r}[C,\Sigma]\,. \tag{A.2}$$

Proof. Let C be a cone based at y and $\Sigma \subseteq \mathbb{R}^n$. Without loss of generality, take $y = 0$. Let $B(0,r) \cap \Sigma \neq \emptyset$. We begin by showing that

$$\mathrm{d}^{0,r}(\Sigma, C) = \widetilde{\mathrm{d}}^{0,r}(\Sigma, C). \tag{A.3}$$

Because $\widetilde{\mathrm{d}}^{0,r}(A,B) \leq \mathrm{d}^{0,r}(A,B)$ for any A and B, it suffices to show that $\mathrm{d}^{0,r}(\Sigma, \mathrm{C}) \leq \widetilde{\mathrm{d}}^{0,r}(\Sigma, \mathrm{C})$. Applying the definition, we must show that for any $x \in \Sigma \cap B(0,r)$, $\mathrm{dist}(x, C \cap B(0,r)) \leq \mathrm{dist}(x, C)$. To do so, we show that if $z \in C$, there is another point $z' \in C \cap B(0,r)$ such that $|x - z'| \leq |x - z|$. Let $x \in \Sigma \cap B(0,r)$ and $z \in C \setminus B(0,r)$. Let $\ell = \{sz : s \in \mathbb{R}\}$ be the line through z passing through 0; note that ℓ is a cone through the origin and that $\ell \subseteq C$ because C is a cone and $z \in C$. Let $\pi : \mathbb{R}^n \to \ell$ be the orthogonal projection onto ℓ. Then the nearest point to x in ℓ is $z' = \pi(x)$. But $|\pi(x)| \leq |x| \leq r$, and so $\pi(x) \in C \cap B(0,r)$. Hence, we have shown (A.3).

We now seek to show that

$$\widetilde{\mathrm{d}}^{0,r}(C, \Sigma) \leq \mathrm{d}^{0,r}(C, \Sigma) \leq 2\,\widetilde{\mathrm{d}}^{0,r}(C, \Sigma). \tag{A.4}$$

Again, we note that the left inequality is automatic. So, by applying the definition, we must show that for any $z \in C \cap B(0,r)$, $\mathrm{dist}(z, \Sigma \cap B(0,r)) \leq 2\,\widetilde{\mathrm{d}}^{0,r}(C, \Sigma)$. Let $\widetilde{d} = \widetilde{\mathrm{d}}^{0,r}(C, \Sigma)$. Note that since $\Sigma \cap B(0,r) \neq \emptyset$ and $0 \in C$, we have that $\widetilde{d} \leq 1$. Let $z \in C \cap B(0,r)$. Then the point $z' = (1 - \widetilde{d})z \in C \cap B(0, r - \widetilde{d}r)$, and $|z - z'| = \widetilde{d}|z| \leq \widetilde{d}\,r$. By assumption, there exists some point $x \in \Sigma$ with $|x - z'| \leq \widetilde{d}\,r$. Thus, we have that $|x - z| \leq |x - z'| + |z' - z| \leq 2\widetilde{d}r$. Further, since $z' \in B(0, r - \widetilde{d}r)$, we have that $|x| \leq |x - z'| + |z'| \leq \widetilde{d}r + (1 - \widetilde{d})r = r$. So $x \in B(0,r)$. Hence,

$$\mathrm{dist}(z, \Sigma \cap B(0,r)) \leq 2\widetilde{d}r. \tag{A.5}$$

Because (A.5) holds for all $z \in B(0,r)$, we have shown (A.4). Thus, we have established (A.1). We conclude by noting that (A.2) follows immediately from (A.1) and the definitions of $D^{0,r}$ and $\widetilde{D}^{0,r}$.

□

Lemma A.1.3. *Let C be a cone based at y and $\Sigma \subseteq \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $r, s > 0$ satisfy $B(y,s) \subseteq B(x,r)$ and $B(y,s) \cap \Sigma \neq \emptyset$. Then*

$$\mathrm{D}^{y,s}[C, \Sigma] \leq 2\frac{r}{s}\,\widetilde{\mathrm{D}}^{x,r}[C, \Sigma]. \tag{A.6}$$

In particular,

$$\mathrm{D}^{y,s}\left[C,\Sigma\right] \leq 2\frac{r}{s}\,\mathrm{D}^{x,r}\left[C,\Sigma\right]. \tag{A.7}$$

Proof. Let all notation and suppositions hold. It follows immediately from the definitions that

$$\widetilde{\mathrm{D}}^{y,s}\left[C,\Sigma\right] \leq \frac{r}{s}\,\widetilde{\mathrm{D}}^{x,r}\left[C,\Sigma\right]. \tag{A.8}$$

We then have that (A.6) follows immediately by Lemma A.1.2. We note that (A.7) follows immediately from (A.6).

□

A.2 Geometry of Planes

In this section we give some basic definitions and lemmas about planes and their geometry. Let V_1 and V_2 be vector spaces with $\dim V_1 \leq \dim V_2$. We define

$$\Gamma(V_1, V_2) = \mathrm{d}^{0,1}\left(V_1, V_2\right). \tag{A.9}$$

It is not hard to see that for any $r > 0$, we have that $\Gamma(V_1, V_2) = \mathrm{d}^{0,r}\left(V_1, V_2\right)$. We extend Γ to a pseudometric on the set of all affine planes. Let P_1 and P_2 be planes $p_1 \in P_1, p_2 \in P_2$ and $\dim P_1 \leq \dim P_2$. Define

$$\Gamma(P_1, P_2) = \Gamma(P_1 - p_1, P_2 - p_2). \tag{A.10}$$

That is, we extend Γ to arbitrary affine planes by first translating them to pass through the origin. Note that this of course does not depend on the p_i. Note that $0 \leq \Gamma(P_1, P_2) \leq 1$. Further, $\Gamma(P_1, P_2) = 0$ if and only if $P_1 || P_2$, and $\Gamma(P_1, P_2) = 1$ if and only if P_1 contains a vector perpendicular to P_2. It is not hard to see that if $P_i^{\perp}$ is an affine orthogonal complement to P_i, then

$$\Gamma(P_2^{\perp}, P_1^{\perp}) = \Gamma(P_1, P_2). \tag{A.11}$$

It will sometimes be convenient to make reference to the angle between two affine planes. We define the angle between P_1 and P_2 to be

$$\measuredangle(P_1, P_2) = \arcsin \Gamma(P_1, P_2). \tag{A.12}$$

Alternatively, we may define the angle between two planes as follows. Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Let $d_{\mathbb{S}}$ denote the path metric on $\mathbb{S}^{n-1}$ defined by $d_{\mathbb{S}}(a,b) = \inf\{\mathrm{len}(\gamma) \mid \gamma : [0,1] \to \mathbb{S}^{n-1}, \gamma(0) = a, \gamma(1) = b\}$. For nonempty sets $A, B \subseteq \mathbb{S}^{n-1}$, we define

$$\mathrm{d}_{\mathbb{S}}(A,B) = \sup_{a\in A}\inf_{b\in B} \mathrm{d}_{\mathbb{S}}(a,b). \tag{A.13}$$

For two vector spaces V_1 and V_2 with $\dim V_1 \leq \dim V_2$, we then have that

$$\measuredangle(V_1,V_2) = d_{\mathbb{S}}(V_1 \cap \mathbb{S}^{n-1}, V_2 \cap \mathbb{S}^{n-1}). \tag{A.14}$$

It follows that for planes P_1 and P_2 with $\dim P_1 \leq \dim P_2$ and $p_i \in P_i$, that

$$\measuredangle(P_1,P_2) = \mathrm{d}_{\mathbb{S}}((P_1 - p_1) \cap \mathbb{S}^{n-1}, (P_2 - p_2) \cap \mathbb{S}^{n-1}) \tag{A.15}$$

Further, from (A.15), subadditivity of angles follows. That is, if P_1, P_2, and P_3 are planes with $\dim(P_1) \leq \dim(P_2) \leq \dim(P_3)$, we get that

$$\measuredangle(P_1,P_3) \leq \measuredangle(P_1,P_2) + \measuredangle(P_2,P_3). \tag{A.16}$$

Further, if $y \in P_1$, $x \in P_2$, then

$$\mathrm{d}^{y,r}(P_1,P_2) \leq 2\left(\measuredangle(P_1,P_2) + \frac{|x-y|}{r}\right). \tag{A.17}$$

Lemma A.2.1. *Let P_1 and P_2, planes in $\mathbb{R}^n$, $\dim P_1 \leq \dim P_2$.*

(1) Let $y \in P_1$, $r > 0$, and $P_2 \cap B(y,r) \neq \emptyset$. Then $\measuredangle(P_1,P_2) \leq \frac{3\pi}{2}\mathrm{d}^{y,r}(P_1,P_2)$.

(2) If $\dim P_1 = \dim P_2 = n-1$ and ν_i is a normal vector to P_i with $\nu_1 \cdot \nu_2 \geq 0$, then $|\nu_1 - \nu_2| \leq \measuredangle(P_1,P_2)$.

Proof. Without loss of generality, take $y = 0$, $r = 1$. We first prove that if both P_1 and P_2 go through 0, then

$$\measuredangle(P_1,P_2) \leq \frac{\pi}{2}\mathrm{d}^{0,1}(P_1,P_2). \tag{A.18}$$

We note that $\arcsin z \leq (\pi/2)z$ for any $z \geq 0$. Hence,

$$\measuredangle(P_1,P_2) = \arcsin(\Gamma(P_1,P_2)) \leq \frac{\pi}{2}\mathrm{d}^{0,1}(P_1,P_2), \tag{A.19}$$

which is (A.18).

Now, suppose that $0 \in P_1$, but that $0 \notin P_2$. Then there exists $p \in P_2$ such that

$$|p| \leq \mathrm{d}^{0,1}(P_1, P_2). \tag{A.20}$$

We compute that

$$\mathrm{d}^{0,1}(P_1, P_2 - p) \leq \mathrm{d}^{0,1}(P_1, P_2) + \mathrm{d}^{0,1}(P_2 - p, P_2). \tag{A.21}$$

Because $P_2 - p$ is a cone through the origin, Lemma A.1.2 tells us that

$$\mathrm{d}^{0,1}(P_2 - p, P_2) \leq 2\,\widetilde{\mathrm{d}}^{0,1}(P_2 - p, P_2) = 2|p| \leq 2\,\mathrm{d}^{0,1}(P_1, P_2). \tag{A.22}$$

Combining (A.21) and (A.22), we get that

$$\mathrm{d}^{0,1}(P_1, P_2 - p) \leq 3\,\mathrm{d}^{0,1}(P_1, P_2). \tag{A.23}$$

Because $P_2 - p$ goes through the origin, we combine (A.18) and (A.23) to get

$$\measuredangle(P_1, P_2) \leq \frac{\pi}{2}\,\mathrm{d}^{0,1}(P_1, P_2 - p) \leq \frac{3\pi}{2}\,\mathrm{d}^{0,1}(P_1, P_2). \tag{A.24}$$

We now prove (2). Without loss of generality, suppose that P_1 and P_2 are codimension 1 planes through the origin. Suppose that for $i = 1, 2$, $\nu_i \perp P_i$, $|\nu_i| = 1$, and that $\nu_1 \cdot \nu_2 \geq 0$. It follows from (A.11) and (A.15) that

$$\measuredangle(P_1, P_2) = \measuredangle(P_2^{\perp}, P_1^{\perp}) = \mathrm{d}_{\mathbb{S}}(\{\pm\nu_2\}, \{\pm\nu_1\}). \tag{A.25}$$

From the fact that $\nu_1 \cdot \nu_2 \geq 0$, it follows that

$$\mathrm{d}_{\mathbb{S}}(\{\pm\nu_2\}, \{\pm\nu_1\}) = \mathrm{d}_{\mathbb{S}}(\nu_2, \nu_1) \geq |\nu_2 - \nu_1|. \tag{A.26}$$

Putting together (A.25) and (A.26), we prove (2). □

A.3 Local Surjectivity of Vector Projections

In this section, we show how to use local set approximation to establish a theorem of local surjectivity of vector projections at flat points of the support. This is similar to the argument given in [9] in the context of flat points. Our version of the theorem will vary, however, in that we seek to establish a version which is quantitative in terms of the scale we consider and the distance to the support. Our main tool will be the Reifenberg Topological Disk Theorem, which we have mentioned before.

We now quote a version of the Reifenberg Topological Disk Theorem which is quantified appropriately for the proof of our main theorem (see for example [9]).

Theorem A.3.1 (Reifenberg Topological Disk Theorem)**.** *There exists $\xi_0 > 0$ and C_0 with the following property. Let $\Sigma \subseteq \mathbb{R}^n$ be a closed set and $y_0 \in \Sigma$. Assume that $r_0 > 0$ and $0 < \xi \leq \xi_0$ satisfy*

$$\Theta_{\Sigma}^{\mathcal{G}}(y, r) \leq \xi \quad \textit{for all } y \in \Sigma \cap B(y_0, 4r_0), 0 < r \leq 10r_0.$$

Let P_0 be a plane through y_0 such that

$$\mathrm{D}^{y_0, 10r_0}\left[P_0, \Sigma\right] \leq \xi.$$

Then there exists a continuous injective map

$$\tau : B(y_0, 3r_0) \cap P_0 \to \Sigma \cap B(0, 4r_0)$$

which satisfies

$$|\tau(y) - y| \leq C_0\, \xi\, r_0 \quad \textit{for all } y \in P_0 \cap B(y_0, 3r_0). \tag{A.27}$$

To provide the quantification necessary in Theorem A.3.4, we provide the following lemmas.

Lemma A.3.2. *For all $\sigma > 0$ and $0 < s \leq 1/2$, there exists $\eta = \eta(\sigma, s)$ with the following property. Let $\Sigma \subseteq \mathbb{R}^4$ be a closed set, $x \in \Sigma$, and C be a KP cone based at x. If $y \in \Sigma$ satisfies*

$$\mathrm{D}^{x, 3|x-y|/2}\left[\Sigma, \mathrm{C}\right] < \eta, \tag{A.28}$$

then

$$\mathrm{D}^{y, s|x-y|}\left[\Sigma, \mathrm{C}\right] < \sigma. \tag{A.29}$$

Proof. Suppose the contrary. Then there exist $\sigma > 0$, $0 < s \leq 1/2$, sequences $x_i, y_i \in \mathbb{R}^4$, KP cones C_i based at x_i, and closed sets Σ_i with $x_i, y_i \in \Sigma_i$ satisfying

$$\mathrm{D}^{x_i, 3|x_i - y_i|/2}\left[\Sigma_i, \mathrm{C}_i\right] < \frac{1}{i} \quad \text{but} \quad \mathrm{D}^{y_i, s|x_i - y_i|}\left[\Sigma_i, \mathrm{C}_i\right] \geq \sigma. \tag{A.30}$$

Let

$$\widetilde{\Sigma}_i = \frac{\Sigma_i - x_i}{|y_i - x_i|}, \widetilde{\mathrm{C}}_i = \frac{\mathrm{C}_i - x_i}{|y_i - x_i|}, \widetilde{y}_i = \frac{y_i - x_i}{|y_i - x_i|}. \tag{A.31}$$

Because $|\widetilde{y}_i| = 1$ we may assume (by applying a rotation) that there exists $y \in \mathbb{R}^4$ with $\widetilde{y}_i = y$ for all i. From (A.30) and (A.31), we have that

$$\mathrm{D}^{0,3/2}\left[\widetilde{\Sigma}_i, \widetilde{\mathrm{C}}_i\right] < \frac{1}{i} \quad \text{but} \quad \mathrm{D}^{y,s}\left[\widetilde{\Sigma}_i, \widetilde{\mathrm{C}}_i\right] \geq \sigma. \tag{A.32}$$

By passing to a subsequence, we may assume that $\widetilde{\Sigma}_i \to \Sigma$ and $\widetilde{C}_i \to C$, where Σ is some closed set and C is a KP cone based at 0. Then (A.32) and Lemma A.1.2 imply that

$$\mathrm{D}^{0,3/2}\left[\Sigma, \mathrm{C}\right] = 0 \quad \text{but} \quad \mathrm{D}^{y,s}\left[\Sigma, \mathrm{C}\right] \geq \sigma > 0. \tag{A.33}$$

Because $s \leq 1/2$ and $|y| = 1$, $B(y, s) \subseteq B(0, 3/2)$. Hence, (A.33) implies that $\Sigma \cap B(0, 3/2) \neq \mathcal{C} \cap B(0, 3/2)$ but $\mathrm{D}^{0,3/2}[\Sigma\mathcal{C}] = 0$, yielding a contradiction.

□

Corollary A.3.3. *For all $\delta > 0$, there exists $A = A(\delta)$, $\varepsilon = \varepsilon(\delta) > 0$ and $\eta = \eta(\delta) > 0$ with the following property. Let $x_0 \in \Sigma \subseteq \mathbb{R}^4$ and $r_0 > 0$. Suppose that*

$$\sup_{\substack{0<r\leq r_0 \\ x\in\Sigma\cap B(x_0,r_0)}} \Theta^{\mathcal{C}}_{\Sigma}(x, r') < \varepsilon \tag{A.34}$$

and C *is a KP cone based at x_0 such that*

$$\mathrm{D}^{x_0,r_0}\left[\Sigma, \mathrm{C}\right] < \eta. \tag{A.35}$$

If $x \in \Sigma$ satisfies $r_0/3 \leq |x - x_0| \leq 2r_0/3$, then for $0 < r \leq |x - x_0|/A$,

$$\Theta^{\mathcal{G}}_{\Sigma}(x, r) < \delta. \tag{A.36}$$

Proof. Let $\delta > 0$ be given. Fix parameters A, ε, $\eta > 0$ to be specified later. Let $x \in \Sigma$ satisfy $r_0/3 \leq |x - x_0| \leq 2r_0/3$. We apply Lemma A.1.3 to (A.35) and get

$$\mathrm{D}^{x_0,2|x-x_0|}\left[\Sigma, \mathrm{C}\right] \leq 2\frac{r_0}{2|x - x_0|}\eta \leq \frac{r_0}{r_0/3}\eta = 3\eta. \tag{A.37}$$

Let $\sigma > 0$ and $s = 1/A$. Then by (A.37), we apply Lemma A.3.2 to get that there exists η small enough so that

$$\mathrm{D}^{x,|x-x_0|/A}\left[\Sigma, \mathrm{C}\right] < \sigma/2. \tag{A.38}$$

By Lemma 4.3.1, we have that for A large enough,

$$\mathrm{D}^{x,|x-x_0|/A}\left[\mathrm{C}, T_x\mathrm{C}\right] < \sigma/2. \tag{A.39}$$

Hence,

$$\mathrm{D}^{x,|x-x_0|/A}\left[\Sigma, T_x\mathrm{C}\right] \leq \sigma. \tag{A.40}$$

We now recall that $\mathcal{G}$ is detectable in $\overline{\mathcal{C}}$. Thus, by Lemma 2.4.5, (A.34) and (A.40) imply that for σ and ε small enough, $\Theta^{\mathcal{G}}_{\Sigma}(x,r) < \delta$ for all $0 < r \leq |x - x_0|/A$.

□

Theorem A.3.4 (Local Surjectivity of Vector Projections)**.** *For all $\delta > 0$, there exist $\varepsilon = \varepsilon(\delta) > 0$ and $\eta = \eta(\delta) > 0$ with the following property. Suppose that $\Sigma \subseteq \mathbb{R}^4$ is a closed set, and*

$$\Theta^{\mathcal{C}}_{\Sigma}(z,r) < \varepsilon \quad \textit{for all } z \in \Sigma \cap B(0,1), 0 < r \leq 1. \tag{A.41}$$

Suppose also that C *is a KP cone based at the origin such that*

$$\mathrm{D}^{0,1}\left[\mathrm{C}, \Sigma\right] < \eta. \tag{A.42}$$

Let $x \in \mathrm{C}$, $|x| = 1/2$, and v be a unit vector such that

$$\measuredangle(v, T_x\mathrm{C}) \geq \frac{\pi}{4}. \tag{A.43}$$

Then there exists $t \in \mathbb{R}$ with $|t| \leq \delta$ and $x + tv \in \Sigma$.

Proof. Let $\delta > 0$ be given. Fix parameters $\varepsilon > 0, \eta > 0$ to be specified later. By (A.41), (A.42), and Corollary A.3.3, for any $\xi > 0$ there exist $\varepsilon > 0$, $\eta > 0$ small enough, and $A > 0$ such that

$$\Theta^{\mathcal{G}}_{\Sigma}(z,r) < \xi \quad \text{ for all } z \in \Sigma, \frac{1}{3} \leq |z| \leq \frac{2}{3}, 0 < r \leq \frac{|z|}{A}. \tag{A.44}$$

Fix $\xi > 0$ to be determined and stipulate that $\varepsilon > 0$ is small enough that (A.44) holds. By (A.42), there exists $y \in \Sigma$ with $|x - y| < \eta$. Stipulate that $\eta \leq 1/12$. Then for all $y' \in B(y, 1/12) \cap \Sigma$, we have that $|y'| \geq |x| - |x - y| - |y' - y| \geq 1/2 - 1/12 - 1/12 = 1/3$. Thus by (A.44),

$$\Theta^{\mathcal{G}}_{\Sigma}(y',r) < \xi \quad \text{ for all } y' \in \Sigma \cap B\left(y, \frac{1}{12}\right), 0 < r \leq \frac{1}{3A}. \tag{A.45}$$

We now require that $\xi \leq \xi_0$ from Theorem A.3.1, $10r_0 \leq 1/(3A)$, and $4r_0 \leq 1/12$. Then (A.45) implies

$$\Theta_{\Sigma}^{\mathcal{G}}(y', r) < \xi \leq \xi_0 \quad \text{for all } y' \in \Sigma \cap B(y, 4r_0), 0 < r \leq 10r_0. \tag{A.46}$$

Statement (A.46) tells us that the hypotheses of Theorem A.3.1 are satisfied. Let P_0 be a plane such that

$$\mathrm{D}^{y,10r_0}\left[P_0, \Sigma\right] < \xi \quad \text{and} \quad y \in P_0. \tag{A.47}$$

Then by Theorem A.3.1, there exists a continuous injective map $\tau : P_0 \cap B(y, 3r_0) \to \Sigma \cap B(y, 4r_0)$ such that

$$|\tau(y') - y'| < C_0\, \xi\, r_0 \quad \text{for all } y' \in P_0 \cap B(y, 3r_0). \tag{A.48}$$

We would now like to know that the angle between v and P_0 is not too small. We seek to establish that

$$\measuredangle(v, P_0) \geq \frac{\pi}{6}. \tag{A.49}$$

To show this, we will first establish that for small enough r_0, η, and ξ,

$$\measuredangle(P_0, T_x\mathrm{C}) \leq \frac{\pi}{12}. \tag{A.50}$$

Recall that $y \in P_0$ by (A.47). So by Lemma A.2.1, it is sufficient to show that

$$\mathrm{d}^{y,3r_0}\left(P_0, T_x\mathrm{C}\right) \leq \frac{1}{18}. \tag{A.51}$$

Let $p \in P_0 \cap B(y, 3r_0)$. Then by (A.47), there exists $z \in \Sigma$ such that

$$|p - z| \leq 10r_0\xi. \tag{A.52}$$

We require that $\xi \leq 1/10$, so that $z \in B(y, 4r_0)$. Recall that $|x - y| \leq \eta$. We require that $\eta \leq r_0$ so that $z \in B(x, 5r_0)$ (actually, we will require later that η be significantly smaller than r_0). Fix $\sigma > 0$ to be specified later. By (A.42) and Lemma A.3.2, we have that

$$\mathrm{D}^{x,5r_0}\left[\Sigma, \mathrm{C}\right] \leq \sigma \tag{A.53}$$

For as η small enough (depending on r_0). Thus, there exists $c \in \mathrm{C} \cap B(x, 5r_0)$ such that

$$|z - c| \leq 5r_0\sigma. \tag{A.54}$$

By Lemma 4.3.1, we have that

$$\mathrm{D}^{x,5r_0}\left[\mathrm{C}, T_x\mathrm{C}\right] \le Cr_0, \tag{A.55}$$

where we use C in this proof to denote a constant which may depend on δ. Thus, there exists a $q \in T_x\mathrm{C}$ such that

$$|c - q| \le Cr_0^2. \tag{A.56}$$

Combining (A.52), (A.54), and (A.56), we get

$$|p - q| \le 10r_0\xi + 5r_0\sigma + Cr_0^2. \tag{A.57}$$

Because for every $p \in P_0 \cap B(y, 3r_0)$, there exists a $q \in T_x\mathrm{C}$ satisfying (A.57), we get that

$$\widetilde{\mathrm{d}}^{y,3r_0}\left(P_0, T_x\mathrm{C}\right) \le \frac{10}{3}\xi + \frac{5}{3}\sigma + Cr_0. \tag{A.58}$$

We now require that

$$\frac{10}{3}\xi, \frac{5}{3}\sigma, Cr_0 \le \frac{1}{108}. \tag{A.59}$$

(That is, we first choose r_0 small enough so that $Cr_0 \le 1/108$, then by Lemma A.3.2, we choose η small enough so that $5/3\sigma \le 1/108$. We also require that $\xi \le 3/1080$.) From (A.58) and (A.59), we get

$$\widetilde{\mathrm{d}}^{y,3r_0}\left(P_0, T_x\mathrm{C}\right) \le \frac{3}{108} = \frac{1}{36}. \tag{A.60}$$

Because P_0 is a cone centered at y, Lemma A.1.2 tells us that

$$\mathrm{d}^{y,3r_0}\left(P_0, T_x\mathrm{C}\right) \le \frac{1}{18}. \tag{A.61}$$

Thus, by previous remarks we have established (A.50) and proven that

$$\measuredangle(P_0, T_x\mathrm{C}) \le \frac{\pi}{12}. \tag{A.62}$$

We conclude that

$$\measuredangle(v, P_0) \ge \measuredangle(v, T_x\mathrm{C}) - \measuredangle(T_x\mathrm{C}, P_0) \ge \frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}. \tag{A.63}$$

For C_0 (still the same constant from (A.48)), we now define the cylinders

$$\begin{aligned} T &= \{x + su + tv : u \in \langle v\rangle^\perp, |u| = 1, |s| \le 3\sin(\pi/6)r_0 + C_0\xi r_0, |t| \le \delta\} \\ T' &= \{x + su + tv : u \in \langle v\rangle^\perp, |u| = 1, |s| \le 3\sin(\pi/6)r_0, |t| \le \delta - C_0\xi r_0\}. \end{aligned} \tag{A.64}$$

We recall that $r_0 \leq 1/12$, and we require that ξ be small enough that

$$C_0 \xi r_0 \leq C_0 \xi \frac{1}{12} \leq \frac{\delta}{2}. \tag{A.65}$$

Recall that $\eta < r_0$ so that in particular, y is in the interior of T'. We now require additionally that

$$\eta, 3\cos(\pi/6) r_0 < \frac{\delta}{4} \tag{A.66}$$

We observe three key facts about the geometry of these cylinders. First,

$$\tau(y') \in \Sigma \cap T \quad \text{ for all } y' \in P_0 \cap T'. \tag{A.67}$$

Second, we observe that by (A.63), (A.66) and the definition of T' (A.64), that $\partial T' \cap P_0$ is an ellipse with minimal axis length at least $3\sin(\pi/6)r_0$. Third, $T' \cap P_0 \subseteq B(y, 3r_0)$. In particular, the map τ is defined on $T' \cap P_0$.

Define $\pi : \mathbb{R}^4 \to P_0$ to be the projection in the v direction onto P_0. Note that to prove the existence of $|t| \leq \delta$ such that $x + tv \in \Sigma$, it suffices to show that $\pi(x)$ has a π-preimage in $T \cap \Sigma$. Suppose the contrary; that is, that for all $y' \in \Sigma \cap T$, $\pi(y') \neq \pi(x)$. Then in particular, consider the continuous map $\pi \circ \tau : T' \cap P_0 \to T \cap P_0$. Because π is projection onto P_0 in the direction of v, we have that $|\pi(z) - z| = \sec(\measuredangle(v, P_0^\perp)) \operatorname{dist}(z, P_0)$ for $z \in \mathbb{R}^4$. Hence, applying and (A.52) and (A.63), for all $y' \in \Sigma \cap B(y, 3r_0) \supseteq \Sigma \cap T$, we have that

$$|\pi(y') - y'| = \sec(\measuredangle(v, P_0^\perp)) \operatorname{dist}(y', P_0) \leq 3r_0 \xi \sec\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \leq C r_0 \xi. \tag{A.68}$$

Coupling this with (A.48), we get that

$$|\pi \circ \tau(y') - y'| \leq C \xi r_0. \tag{A.69}$$

By assumption, $\pi \circ \tau$ misses $\pi(x)$, so we may define a continuous retract $h : P_0 \backslash \{\pi(x)\} \cap T' \to P_0 \cap \partial T'$ which fixes $P_0 \cap \partial T'$ (e.g. radial projection). Thus, we create a continuous map $\varphi = h \circ \pi \circ \tau : P_0 \cap T' \to P_0 \cap \partial T'$ such that

$$|\varphi(y') - y'| \leq C \xi r_0 \quad \text{ for all } y' \in P_0 \cap \partial T'. \tag{A.70}$$

But because $P_0 \cap T'$ is an ellipse with minimal axis of length at least $3\sin(\pi/6)r_0$, φ restricted to $P_0 \cap \partial T'$ has degree 1 for small enough ξ. However, by degree theory for the sphere, a continuous map on the sphere extends continuously over the ball if and only if the degree of the map is 0. So for ξ small enough, we get a contradiction. Hence for small enough ξ, $\pi(x)$ has a preimage in T and the lemma is proven. $\square$

Appendix B

LOCAL COORDINATES ON THE KP CONE

B.1 Construction and analysis of coordinates

In this Section, we give the technical details of some lemmas about the behavior of approximability under diffeomorphisms.

Lemma B.1.1. *Suppose that $U, V \subseteq \mathbb{R}^n$ are open sets, $\Gamma \subseteq \mathbb{R}^n$, $0 < m < n$, and $\psi \in C^2(U, V)$ is bijective and satisfies*

$$0 < \lambda \leq \frac{|\psi(x) - \psi(y)|}{|x - y|} \leq \Lambda \text{ for all } x, y \in U \quad \text{and} \quad ||D^2\psi||_\infty = \sup_{x \in U} ||D^2_x \psi|| < \infty. \quad \text{(B.1)}$$

Let $z \in \Gamma \cap U$, $B(z, r) \subseteq U$, P be a plane of dimension m through z, and set $\widetilde{P} = D_z\psi(P - z) + \psi(z)$, $\widetilde{\Gamma} = \psi(\Gamma)$. Then

$$\mathrm{d}^{\psi(z), \lambda r}\left(\widetilde{\Gamma}, \widetilde{P}\right) \leq \frac{||D^2\psi||_\infty}{2\lambda} r + \frac{\Lambda}{\lambda} \mathrm{d}^{z,r}(\Gamma, P). \quad \text{(B.2)}$$

Proof. Without loss of generality, take $z = \psi(z) = 0$. Let $P \subseteq \mathbb{R}^n$ be an m-plane through 0 and set $d = \mathrm{d}^{0,r}(\Gamma, P)$. Note that $\lambda \leq \frac{|D_0\psi(v)|}{|v|} \leq \Lambda$ for all $v \in \mathbb{R}^n \setminus \{0\}$ by (B.1). Further, note that we have

$$B(0, \lambda r) \subseteq \psi(B(0, r)). \quad \text{(B.3)}$$

Let $y \in \widetilde{\Gamma} \cap B(0, \lambda r)$. Then by (B.3) and bijectivity, we have that there exists $x \in B(0, r) \cap \Gamma$ such that $y = \psi(x)$. By $d = \mathrm{d}^{0,r}(\Gamma, P)$, we get that there exists $p \in P$ such that $|p - x| \leq rd$. Let $\widetilde{p} = D_0\psi(p) \in \widetilde{P}$. We compute

$$|\widetilde{p} - y| = |D_0\psi(p) - \psi(x)| \leq |D_0\psi(p) - \psi(p)| + |\psi(p) - \psi(x)|. \quad \text{(B.4)}$$

By (B.1) and Taylor expansion, we get that

$$|\widetilde{p} - y| \leq \frac{||D^2\psi||_\infty}{2} |p - 0|^2 + \Lambda |p - x| \leq \frac{||D^2\psi||_\infty}{2} r^2 + \Lambda r d. \quad \text{(B.5)}$$

Because for every $y \in B(0,\lambda r) \cap \widetilde{\Gamma}$, there exists $\widetilde{p} \in \widetilde{P}$ satisfying (B.5), we get that

$$\widetilde{\mathrm{d}}^{0,\lambda r}\left(\widetilde{\Gamma}, \widetilde{P}\right) \leq \frac{||D^2\psi||_\infty}{2\lambda} r + \frac{\Lambda}{\lambda} d. \tag{B.6}$$

By Lemma A.1.2, we get that (B.6) tells us (B.2). □

Lemma B.1.2. *For $a \in \mathrm{C} \setminus \{0\}$, and $A > 0$ large enough, there exists a neighborhood $U \supseteq B(a, 2|a|/A)$, $V \subseteq \mathbb{R}^3$ open, $I \ni 0$ an open interval, and a smooth coordinate map $\psi^a : U \to V \times I$ such that $V \times \{0\} = \psi^a(\mathrm{C} \cap U)$ and $\widetilde{\pi} = \psi^a \circ \pi \circ (\psi^a)^{-1}$ is orthogonal projection onto $\mathbb{R}^3 \times \{0\}$ (where π is the same map defined in Section 5; see (5.93)). Further, ψ^a satisifes the estimates*

$$\frac{1}{2} \leq \frac{|\psi^a(x) - \psi^a(y)|}{|x-y|} \leq 2 \qquad \text{for all } x, y \in U \tag{B.7}$$

and

$$||D^2\psi^a||_\infty = \sup_{x \in U} ||D^2_x \psi^a|| \leq \frac{C}{|a|} \tag{B.8}$$

for some C independent of a.

Proof. First, we fix an $a \in \mathrm{C}$, $|a| = 1$. We define ψ^a by defining its inverse. Choose orthonormal coordinates (z_1, z_2, z_3) on $T_a\mathrm{C}$ centered at a. Let p be orthogonal projection from C onto $T_a\mathrm{C}$, and take $U' \supseteq B(a, 8/A)$ to be an open set where p^{-1} is defined. Identifying $T_a\mathrm{C}$ with $\mathbb{R}^3$ under the z coordinates, let $V = U' \cap T_a\mathrm{C}$. Let $I = (-8/A, 8/A)$. Define for $(z,t) \in V \times I$

$$(\psi^a)^{-1}(z,t) = p^{-1}(z) + t\eta_{p^{-1}(z)}. \tag{B.9}$$

Assume that $A \geq 16$, so that $8/A \leq 1/2$. Note that ψ^a is bijective onto $U = (\psi^a)^{-1}(V \times I)$, because η is a smooth vector field (away from the x_4 axis), and the point (z,t) is the flow after time t of the point $p^{-1}z$ along the integral curves of η. Further, the same comments imply that it is smooth. Then we note that because $p^{-1}(z) \in \mathrm{C}$, $V \times \{0\} = \psi^a(\mathrm{C} \cap U)$. Further, $\widetilde{\pi}(z,t) = \psi^a(\pi(p^{-1}(z) + t\eta_{p^{-1}(z)})) = \psi^a(p^{-1}(z)) = z$, and so $\widetilde{\pi}$ is orthogonal projection onto $V \times \{0\}$.

We now show that (B.7) holds for ψ^a as long as U' is chosen small enough and $1/A$ is chosen small enough. Continuing the identifcation of $T_a\mathrm{C}$ with $\mathbb{R}^3$, we set e_i to be the coordinate vector of z_i, and note that $D_a\psi^a$ is the map

$$D_a\psi^a(e_i) = e_i, \quad D_a\psi^a\eta_a = e_4 \qquad \text{for } i = 1, 2, 3. \tag{B.10}$$

Because the z_i are orthonormal and $\measuredangle(\eta_a, T_a\mathrm{C}) = \pi/4$, we get that

$$\langle e_i, e_j\rangle = \delta_{ij}, \quad |\langle e_i, \eta_a\rangle| \leq 1/\sqrt{2}. \tag{B.11}$$

From (B.10) and (B.11), as well as recalling that $|\nu_a| = 1$, we get that

$$\frac{1}{\sqrt{2}} \leq \frac{|D_a\psi^a v|}{|v|} \leq \sqrt{2} \quad \text{for } v \in \mathbb{R}^4 \setminus \{0\}. \tag{B.12}$$

Because ψ^a is smooth, it follows from (B.12) that for U' small enough,

$$\frac{1}{2} \leq \frac{|\psi^a(x) - \psi^a(y)|}{|x - y|} \leq 2 \quad \text{for } x, y \in U'. \tag{B.13}$$

Thus if A is large enough, $B(a, 8/A) \subseteq U'$. By definition of U and (B.13), we have that $B(a, 2/A) \subseteq U$. Because ψ is C^2, by potentially restricting to a compactly contained open set, we may assume

$$C = ||D^2\psi^a||_\infty < \infty. \tag{B.14}$$

Let $b \in \mathrm{C}$, $|b| = 1$. Then there is a rotation $O \in O(4)$ taking b to a and fixing C. Define $\psi^b = O^{-1} \circ \psi^a \circ O$. For $b \in \mathrm{C} \setminus \{0\}$, we define $\psi^b = |b|\psi^{\frac{b}{|b|}}(\cdot/|b|)$. Note that $\psi^{\frac{b}{|b|}}$ satisfies $(B.13)$ and $(B.14)$, we have that ψ^b satisfies $(B.7)$ and $(B.8)$.

□

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