Key

QUIZ 5

MATH 4242 010, AU'14

Please write your **name on the top left** and show all work legibly.

Problem 1. Let $v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$. Do these vectors form a basis of $\mathbb{R}^4$? Explain your reasoning.

Consider the matrix $A = (v_1\ v_2\ v_3\ v_4)$. Then $\text{rng}(A) = \text{span}(v_1, v_2, v_3, v_4)$. Performing Gaussian Elimination, we find the row echelon form of A below.

$$\begin{pmatrix} 0 & -3 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & -3 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 7/3 \end{pmatrix}.$$

Thus, $\text{rank}(A) = 4$ and we conclude that $\dim(\text{span}(v_1\ v_2\ v_3\ v_4)) = 4$ so $\text{span}\{v_1 - v_4\} = \mathbb{R}^4$. Because $\dim(\mathbb{R}^4) = 4$, v_1, v_2, v_3, v_4 is a basis.

Argument 2 (Alternative)

As above, A has a row echelon form of $\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 7/3 \end{pmatrix} = U$.

Thus, the method of back substitution will always yield a solution since U is upper triangular with nonzero entries on the diagonal. We conclude that any b will yield a solution to $Ax = b$ so $\text{span}(v_1\ v_2\ v_3\ v_4) = \mathbb{R}^4$. Because $\dim(\mathbb{R}^4) = 4$, v_1, v_2, v_3, v_4 is a basis