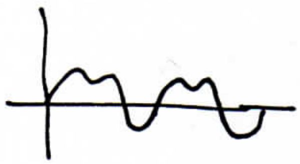
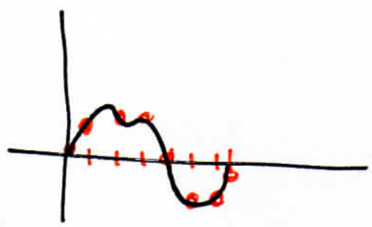
Lecture 32 §5.7 The discrete fourier transform

Suppose we are trying to understand a continuous periodic signal



We will try to "understand" the signal. Computationally, we can only store, read, and work with a finite number of data points. We will call these the sample points, indicated in orange



Our goal is to understand the frequencies in terms of the sampled values. This is known as Discrete Fourier Analysis, and opens the door to acoustic manipulation, noise removal, and many other topics.

Setup

We will assume our interval of periodicity is $0 \leq x \leq 2\pi$. We sample at the n sample points,

$$x_0 = 0, \quad x_1 = \frac{2\pi}{n}, \quad x_2 = \frac{4\pi}{n}, \quad \dots, \quad x_k = \frac{2k\pi}{n}, \quad \dots, \quad x_{n-1} = \frac{2(n-1)\pi}{n}.$$

Let $f(x)$ be the true signal. Our goal is to reconstruct f from the sampled values. Let $\vec{f}$ be the sampled vector,

$$\vec{f} = \left(f(x_0), f(x_1), f(x_2), \dots, f(x_{n-1}) \right).$$

We will consider f which are either real or complex, but our techniques will be complex in either case.

Recall that $e^{ix} = \cos x + i\sin x$, So also $e^{i2x} = \cos 2x + i\sin 2x$,

$e^{i3x} = \cos 3x + i\sin 3x \ldots, e^{ikx} = \cos kx + i\sin kx.$

Because $\cos kx$ & $\sin kx$ are 2π periodic, so is e^{ikx}.

Moreover, it turns out that any 2π periodic function $f(x)$ can be written as an infinite sum

$$f(x) = \sum_{k=0}^{\infty} c_k e^{ikx}, \quad \text{for some constants } c_k.$$

Our technique is thus to consider only the finite sum

$$f(x) \approx \sum_{k=0}^{n-1} c_k e^{ikx}, \quad \text{for some constants } c_k,$$

where we've chosen the first n terms because we have n sample points $x_0 - x_{n-1}$. This will allow us to solve the appropriate problem.

Just as we have a sample of f, we will work with samples of the functions e^{ikx}. So we set w_k to be the sample of e^{ikx}

$$w_k = \left(e^{ikx_0}, e^{ikx_1}, \ldots, e^{ikx_{n-1}}\right) = \left(e^0, e^{\frac{2k\pi}{n}i}, \ldots, e^{\frac{2k\pi}{n}(n-1)i}\right).$$

Set $\zeta_n = e^{\frac{2\pi}{n}i}$. Then we see

$$w_k = \left(\zeta_n^{0\cdot k}, \zeta_n^{1\cdot k}, \zeta_n^{2\cdot k}, \ldots, \zeta_n^{(n-1)k}\right).$$

ex For $n = 4$, $\zeta_4 = e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = i$

$$w_0 = (\zeta_4^0, \zeta_4^0, \zeta_4^0, \zeta_4^0) = (1, 1, 1, 1)$$

$$w_1 = (i^0, i^1, i^2, i^3) = (1, i, -1, -i)$$

$$w_2 = (i^0, i^2, i^4, i^6) = (1, -1, 1, -1)$$

$$w_3 = (i^0, i^3, i^6, i^9) = (1, -i, -1, i)$$

We will choose our constants c_k so that our sample exponentials sum matches our sample vector $\vec{f}$,

$$\vec{f} = \sum_{k=0}^{n-1} c_k \omega_k .$$

This is possible because $\vec{f}, \omega_0, \omega_1, \dots, \omega_{n-1} \in \mathbb{C}^n$, and $\omega_0 \dots \omega_{n-1}$ form a basis of $\mathbb{C}^n$. In fact, the situation is even better than that.

Defn The <u>averaged dot product</u> on $\mathbb{C}^n$ is

$$\langle u, v \rangle = \frac{1}{n} u \cdot v = \frac{1}{n} \sum_{k=0}^{n-1} u_k \overline{v_k} .$$

We will work with the averaged dot product for the rest of this section. Our sampled exponentials vectors have the following helpful property

Fact $\omega_0 \dots \omega_{n-1}$ form an orthonormal basis of $\mathbb{C}^n$ with the averaged dot product. That is,

$$\langle \omega_j, \omega_k \rangle = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} .$$

Moreover, this implies the complex analogue of Thm 5.7, that

$$\vec{f} = \sum_{k=0}^{n-1} c_k \omega_k \quad \text{where} \quad c_k = \langle \vec{f}, \omega_k \rangle .$$