§5.1 Orthogonal and Orthonormal Bases Lecture 29

Let $(V, \langle , \rangle)$ be an inner product space.

Defn An <u>orthogonal basis of V</u> is a basis $u_1 \ldots u_n$ such that $\langle u_i, u_j \rangle = 0$ for $i \neq j$. An <u>orthonormal basis of V</u> is an orthogonal basis $u_1 \ldots u_n$ such that $\|u_i\| = 1$ for all i.

ex The simplest and most familiar example is the standard basis of $\mathbb{R}^n$ with the dot product,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \text{ because } e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

ex Consider $\mathbb{R}^n$ with the weighted inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} w_i x_i y_i \quad \text{for weights } w_i > 0.$$

Then $\frac{1}{\sqrt{w_1}} e_1, \frac{1}{\sqrt{w_2}} e_2, \ldots, \frac{1}{\sqrt{w_n}} e_n$ is an orthonormal basis,

because

$$\langle e_i, e_j \rangle = \begin{cases} 0 & i \neq j \\ w_i \cdot \frac{1}{\sqrt{w_i}^2} & i = j \end{cases} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

ex Consider $\mathbb{R}^4$ with the dot product. Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

This is an orthogonal basis. Setting $u_i = \frac{v_i}{\|v_i\|}$, we get that

$$\|u_i\| = \left\| \frac{v_i}{\|v_i\|} \right\| = \frac{\|v_i\|}{\|v_i\|} = 1.$$

So u_i form an orthogonal basis

Computations in orthonormal bases are very nice. Consider an arbitrary basis, not necessarily orthogonal. In order to express

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n,$$

we must form the system

$$(v_1 v_2 \dots v_n \mid v)$$

and solve for the coefficient vector c. However in an orthonormal basis, we have the following.

Theorem 5.7 Let $(V, \langle , \rangle)$ be an inner product space, ~~and~~ $v \in V$, ~~Then in these coordinates~~, and $v_1 \dots v_n$ be an orthonormal basis. Then

$$v = c_1 v_1 + \dots + c_n v_n$$

where

$$c_i = \langle v, v_i \rangle.$$

That is,

$$v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i \qquad \text{for any } v \in V.$$

Moreover,

$$\| v \| = \sqrt{\sum_{i=1}^{n} c_i^2}.$$

Pf Let $v_1, \dots v_n$ be an orthonormal basis for an inner product space $(V, \langle , \rangle)$. Since $v_1, \dots v_n$ is a basis, there are constants $c_1, \dots, c_n \in \mathbb{R}$ such that

$$v = c_1 v_1 + \dots + c_n v_n.$$

We compute

$$\langle v, v_i \rangle = \langle \sum_{j=1}^{n} c_j v_j, v_i \rangle \underset{\text{linearity}}{=} \sum_{j=1}^{n} c_j \langle v_j, v_i \rangle.$$

Because $v_1 \dots v_n$ are orthonormal, $\langle v_j, v_i \rangle = 0$ for $j \neq i$, and $\langle v_j, v_i \rangle = 1$ when $j = i$. Thus,

$$\langle v, v_i \rangle = \sum_{j=1}^{n} c_j \langle v_j, v_i \rangle = c_i,$$

establishing the first part.

To see the second part, we compute

$$(1) \quad \|v\|^2 = \langle v, v\rangle = \left\langle \sum_{i=1}^{n} c_i v_i, \sum_{j=1}^{n} c_j v_j \right\rangle \underset{\substack{\uparrow \\ \text{bilinearity}}}{=} \sum_{i=1}^{n} c_i \sum_{j=1}^{n} c_j \langle v_i, v_j\rangle .$$

For each i, $\langle v_i, v_j\rangle = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases}$, thus

$$c_i \sum_{j=1}^{n} c_j \langle v_i, v_j\rangle = c_i \cdot c_i = c_i^2 .$$

Substituting into (1),

$$\|v\|^2 = \sum_{i=1}^{n} c_i^2 ,$$

proving the second part.