Lecture 28 Polynomial and General Function data fitting

Our data fitting technique can be easily generalized from fitting a line to fitting a general polynomial.

Again, suppose we are given m data points $(t_1, y_1) \dots (t_m, y_m)$. We attempt to fit a polynomial

$$y(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_n t^n$$

Similarly to before, define

$$A = \begin{pmatrix} 1 & t_1 & & t_1^n \\ 1 & t_2 & \cdots & t_2^n \\ \vdots & \vdots & & \vdots \\ 1 & t_m & & t_m^n \end{pmatrix}, \quad x = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

So that

$$Ax = \begin{pmatrix} \alpha_0 + \alpha_1 t_1 + \dots + \alpha_n t_1^n \\ \alpha_0 + \alpha_1 t_2 + \dots + \alpha_n t_2^n \\ \vdots \\ \alpha_0 + \alpha_1 t_m + \dots + \alpha_n t_m^n \end{pmatrix} = \begin{pmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_m) \end{pmatrix}.$$

Let

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

The error vector between the polynomial $y(t)$ and the sampled points $y_1 \dots y_m$ is

$$Ax - \vec{y}$$

so, as before, we strive to minimize the norm of the error vector

$$\|Ax - \vec{y}\|.$$

So we must find the least squares solution to the system $Ax = \vec{y}$.

Fact The matrix A has $\ker(A) = \{0\}$ so long as $n+1 \leq m$ and $t_1 \dots t_m$ are distinct.

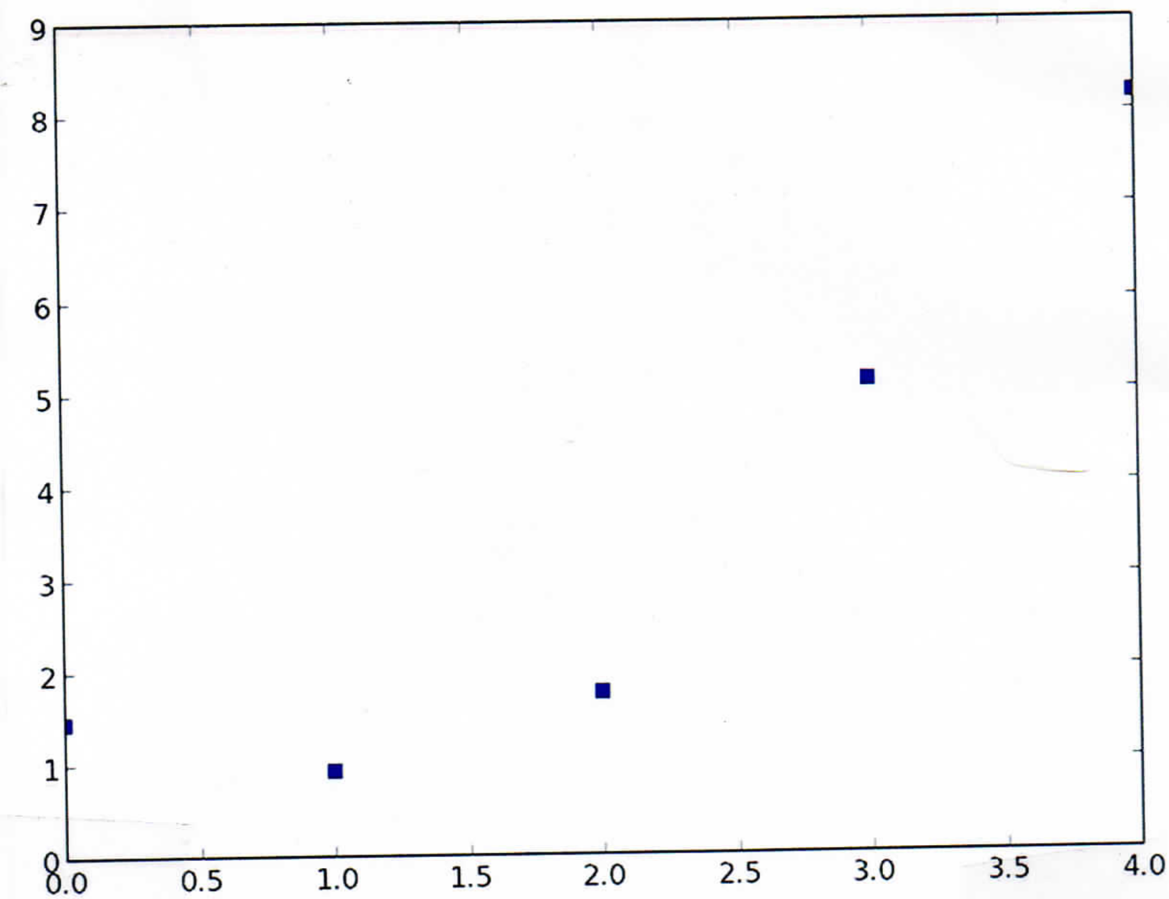
Thus, we may apply Thm 4.8, and we get that the least squaes solution x^* is given as the solution to the normal equations

$$A^T A x = A^T \vec{y} .$$

ex Suppose we are given data points

t_i	0	1	2	3	4
y_i	1.45	.93	1.76	5.11	8.19

We plot these data points and see the following



From the graph, we can guess that a quadratic polynomial will fit these points well. That is, we try to fit a polynomial of the form

$$y(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$$

to the data

So, as before, we set

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1.45 \\ .93 \\ 1.76 \\ 5.11 \\ 8.19 \end{pmatrix}, \quad x = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

To solve the normal equations, we find

$$A^T A = \begin{pmatrix} 5 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{pmatrix} \qquad A^T \vec{y} = \begin{pmatrix} 17.45 \\ 52.55 \\ 185.03 \end{pmatrix}.$$

Solving the normal equations

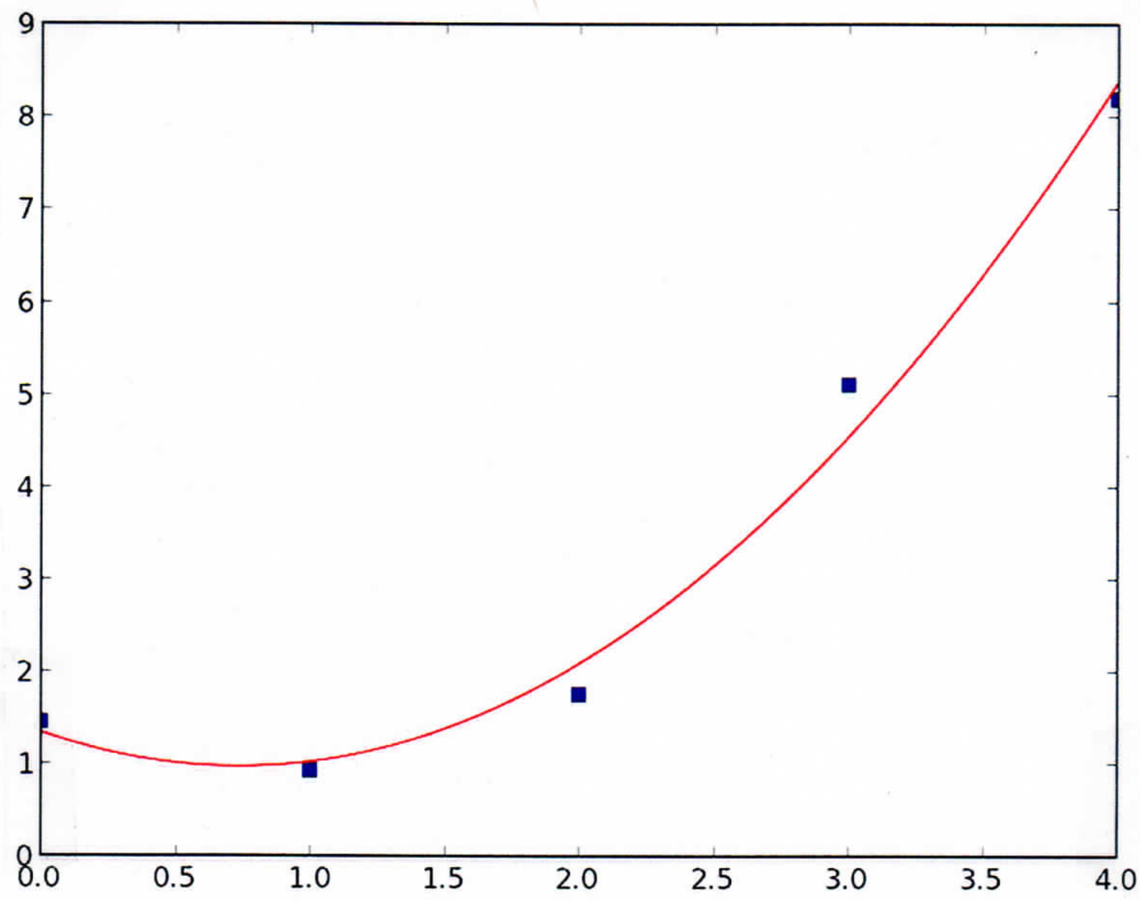
$$A^T A\, x = A^T \vec{y}$$

yields

$$x = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1.350 \\ -1.015 \\ 0.695 \end{pmatrix},$$

So our degree 2 polynomial of best fit is

$$y(t) = 1.350 - 1.015\,t + .695\,t^2$$



Interpolation.

In the case $m=n+1$, A is a square matrix with $\ker(A)=\{0\}$. Thus, A is nonsingular, so

$$A x = \vec{y}$$

can be solved exactly. That is, there is a unique degree n polynomial which passes through $n+1$ data points. This is called the interpolating polynomial.

ex Find the interpolating polynomial passing through $\cos(t)$ at $t = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

Our data points are

t_i	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\cos(t_i)$	1	$\frac{1}{\sqrt{2}}$	0

Since we have 3 points, the interpolating polynomial is degree 2. So we form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{\pi}{4} & \frac{\pi^2}{16} \\ 1 & \frac{\pi}{2} & \frac{\pi^2}{4} \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Since A is square, nonsingular, $Ax = \vec{y}$ can be solved exactly as

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & \frac{\pi}{4} & \frac{\pi^2}{16} & \frac{1}{\sqrt{2}} \\ 1 & \frac{\pi}{2} & \frac{\pi^2}{4} & 0 \end{array}\right) \quad \begin{matrix} r_2 - r_1 \\ r_3 - r_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & \frac{\pi}{4} & \frac{\pi^2}{16} & \frac{1}{\sqrt{2}} - 1 \\ 0 & \frac{\pi}{2} & \frac{\pi^2}{4} & -1 \end{array}\right)$$

$$r_3 - 2r_2 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & \frac{\pi}{4} & \frac{\pi^2}{16} & \frac{1}{\sqrt{2}} - 1 \\ 0 & 0 & \frac{\pi^2}{8} & 1 - \sqrt{2} \end{array}\right) \rightsquigarrow \boxed{\alpha_0 = 1}$$

$$\boxed{\alpha_2 = \frac{(1-\sqrt{2})\,8}{\pi^2}}$$

$$\frac{\pi}{4}\alpha_1 = -\frac{\pi^2}{16}\alpha_2 + \frac{1}{\sqrt{2}} - 1 = -\frac{1-\sqrt{2}}{2} + \frac{\sqrt{2}-2}{2} = \frac{2\sqrt{2}-3}{2}$$

$$\alpha_1 = \frac{2}{\pi}(2\sqrt{2} - 3)$$

In fact, this techinique can be extended to any function of the form

$$y(t) = \alpha_0 g_0(t) + \alpha_1 g_1(t) + \ldots + \alpha_n g_n(t).$$

A common example of this is

$$(1) \qquad y(t) = \alpha_0 + \alpha_1 \cos(t) + \alpha_2 \sin(t).$$

So, as before, we set

$$A = \begin{pmatrix} g_0(t_1) & g_1(t_1) & & g_n(t_1) \\ g_0(t_2) & g_1(t_2) & \cdots & g_n(t_2) \\ \vdots & \vdots & & \vdots \\ g_0(t_m) & g_1(t_m) & & g_n(t_m) \end{pmatrix},$$

e.g. if we take y as in (1),

$$A = \begin{pmatrix} 1 & \cos(t_1) & \sin(t_1) \\ 1 & \cos(t_2) & \sin(t_2) \\ \vdots & \vdots & \vdots \\ 1 & \cos(t_m) & \sin(t_m) \end{pmatrix}.$$

For a general class of functions, there is no guarantee that A will have $\ker(A) = \{0\}$; in applications, this should be checked.