Lecture 25 More on the nearest point problem

In these notes, we discuss more on solving the nearest point problem. First, we recall the problem & notation. Given an inner product $\langle , \rangle$ on $\mathbb{R}^m$, a subspace $V \subseteq \mathbb{R}^m$, & a point $b \in \mathbb{R}^m$, find the nearest point in V to b; that is, find $v \in V$ which minimizes the associated norm $\|v - b\|$.

To this end, we take a basis $v_1 \dots v_n$ of V and ~~let~~ write a general $v \in V$ as

$$v = x_1 v_1 + \dots + x_n v_n = \sum_{i=1}^{n} x_i v_i, \qquad x_i \in \mathbb{R}.$$

In these coordinates, we found

$$\|v - b\|^2 = \langle v, v \rangle - 2\langle v, b \rangle + \langle b, b \rangle = x^T K x - 2x^T f + c =: p(x),$$

where $K = (k_{ij})$, $k_{ij} = \langle v_i, v_j \rangle$, $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, $f_i = \langle v_i, b \rangle$, $c = \|b\|^2$. Moreover, $K > 0$ by Theorem 3.28.

Now we discuss finding K & f efficiently.

Case 1: $\langle , \rangle$ = dot product

In this case, $k_{ij} = v_i \cdot v_j = v_i^T v_j$. Let $A = (v_1 \dots v_n)$. Then we note that ~~the~~ by definition

$$(A^T A)_{ij} = \sum_{k=1}^{m} (A^T)_{ik} (A)_{kj} = \sum_{k=1}^{m} (v_i)_k (v_j)_k = v_i^T v_j = k_{ij}.$$

So $K = A^T A$.

Let's compute f. $f_i = v_i \cdot b = v_i^T b$, so similarly $f = A^T b$.

General Case: By Theorem 3.21, a general inner product on $\mathbb{R}^m$ is given by $\langle x, y \rangle = x^T C y$ for some positive definite matrix C.

In this case,

$$k_{ij} = \langle v_i, v_j \rangle = v_i^T C v_j .$$

Letting $A = (v_1 \dots v_n)$, we find as before that

$$K = A^T C A .$$

Similarly,

$$f_i = \langle v_i, b \rangle = v_i^T C b$$

So

$$f = A^T C b.$$

Practically speaking, this means it is best to compute $A^T C$, then compute $K = (A^T C) A$ & $f = (A^T C) b$.

ex Let $C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$, and $\langle x, y \rangle = x^T C y$. Because $C > 0$, $\langle , \rangle$ forms an inner product.

Problem Find the nearest point in $V = \text{span}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}\right)$ to $b = \begin{pmatrix}1\\2\\3\end{pmatrix}$.

Solution Set $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$. We compute

$$A^T C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & -2 & -4 \end{pmatrix},$$

$$K = A^T C A = \begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix}, \quad f = A^T C b = \begin{pmatrix} 10 \\ -14 \end{pmatrix}.$$

We find x^*, the minimizer of $p(x)$ by solving

$$Kx = f. \quad \left(\begin{array}{cc|c} 4 & -4 & 10 \\ -4 & 8 & -14 \end{array}\right) \xrightarrow{r_2 + r_1} \left(\begin{array}{cc|c} 4 & -4 & 10 \\ 0 & 4 & -4 \end{array}\right), \quad x_2^* = -1, \quad x_1^* = x_2^* + \tfrac{5}{2} = \tfrac{3}{2}.$$

So the nearest point is

$$v^* = x_1^* v_1 + x_2^* v_2 = \frac{3}{2}\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{5}{2} \\ \frac{5}{2}\end{pmatrix}.$$

The distance from v^* to b can be computed by Theorem 4.5,

$\| v^* - b \| = \sqrt{\|b\|^2 - f^T x^*}$. So we compute

$$\|b\|^2 = \langle b, b \rangle = b^T C b = (1\ 2\ 3) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (-2\ 4\ 8) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= -2 + 8 + 24 = 30$$

$$f^T x^* = (10\ \ -14) \begin{pmatrix} \frac{3}{2} \\ -1 \end{pmatrix} = 15 + 14 = 29,$$

and find the minimum distance to be

$$\| v^* - b \| = \sqrt{30 - 29} = 1.$$