Lecture 24 §4.3 Nearest Point Problem

In these notes, we fix an inner product $\langle , \rangle$ on $\mathbb{R}^n$ and let $\|\cdot\|$ be its associated norm. We now study the "nearest point problem". That is, give a subspace V and a point $b \in \mathbb{R}^n$, find the vector $v \in V$ which is the nearest point to b. Or said another way, we look for the vector $v^* \in V$ which minimizes the distance $\|v - b\|$.

Observation 1 Notice that if $b \in V$, $v^* = b$ is the minimizer because $\|b - b\| = \|0\| = 0$.

Let $v_1, \ldots v_n \in V$ form a basis of V. ~~What~~ We express a general vector $v \in V$ as

$$v = x_1 v_1 + \ldots + x_n v_n = \sum_{i=1}^{n} x_i v_i .$$

for real numbers x_i.

Observation 2 Notice that since our goal is to minimize $\|v - b\|$ which is nonegative, we may equivalently minimize $\|v - b\|^2$. We compute

$$(1) \quad \|v - b\|^2 = \langle v - b, v - b \rangle = \|v\|^2 - 2\langle v, b\rangle + \|b\|^2$$

We now expand (1). First, we compute

$$\|v\|^2 = \left\langle \sum_{i=1}^{n} x_i v_i , \sum_{j=1}^{n} x_j v_j \right\rangle = \sum_{i,j=1}^{n} x_i x_j \langle v_i, v_j \rangle .$$

Observation 3 The matrix K with coefficients $k_{ij} = \langle v_i, v_j \rangle$ is symmetric and positive definite. (See theorem 3.28; this matrix is p.d. because $v_1 \ldots v_n$ are linearly independent).

$$(2) \quad \text{So} \quad \|v\|^2 = x^T K x \quad \text{for } K > 0 .$$

Next, we expand $\langle v, b \rangle = \left\langle \sum_{i=1}^{n} x_i v_i , b \right\rangle = \sum_{i=1}^{n} x_i \langle v_i, b \rangle = \sum_{i=1}^{n} x_i f_i$ where $f_i = \langle v_i, b \rangle$. So

$$(3) \quad -2\langle v, b \rangle = -2 x^T f .$$

Combining (1), (2) & (3), we have

$$(4) \quad \|v-b\|^2 = x^T K x - 2x^T f + \|b\|^2.$$

Since b is fixed, this expression defines a quadratic polynomial in x in symmetric form. Thus, using Thm 4.1, we conclude the following.

Thm 4.5 Let $v_1, \dots v_n$ form a basis of a subspace $V \subseteq \mathbb{R}^m$. Given $b \in \mathbb{R}^m$, the unique minimizer of $\|v-b\|$ is given by

$$v^* = x_1^* v_1 + \dots + x_n^* v_n,$$

where

$$x^* = \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix} = K^{-1} f, \text{ where } k_{ij} = \langle v_i, v_j \rangle \text{ and } f_i = \langle v_i, b \rangle.$$

The minimum distance is

$$d^* = \|v^* - b\| = \sqrt{\|b\|^2 - f^T x^*}.$$

Pf As observed in (4), the function

$$p(x) = x^T K x - 2x^T f + \|b\|^2$$

is a quadratic function in symmetric form. By Theorem 4.1, its unique minimizer is

$$x^* = K^{-1} f.$$

So the unique minimizer of $\|v-b\|$ is

$$v^* = \sum_{i=1}^{n} x_i v_i$$

and

$$p(x^*) = \|v^* - b\|^2 = \|b\|^2 - f^T x^*$$

by Thm 4.1. //

ex Consider the plane given by $2x_1 + x_2 - x_3 = 0$ in $\mathbb{R}^3$ with the dot product. This plane is the set $\{x : 2x_1 + x_2 - x_3 = 0\} = \{x : (2\ 1\ {-1})\, x = 0\}$
$= \operatorname{Ker}((2\ 1\ {-1}))$.

Hence, we compute a basis for it by analyzing the free variables. Setting $x_2 = 1$, $x_3 = 0$, we find $x_1 = -\frac{1}{2}$. Setting $x_2 = 0$ $x_3 = 1$, we find $x_1 = \frac{1}{2}$. So we get basis vectors

$$\tilde{v}_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \quad \tilde{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}.$$

For convenience, lets work with

$$v_1 = 2\tilde{v}_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \quad v_2 = 2\tilde{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Then we compute $\langle v_1, v_1 \rangle = 5$, $\langle v_1, v_2 \rangle = -1$, $\langle v_2, v_2 \rangle = 5$, so

$$K = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

Suppose we search for the nearest point to $b = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$. Then

$$f = \begin{pmatrix} \langle v_1, b \rangle \\ \langle v_2, b \rangle \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

Then x^* is given by solving $Kx = f$.

$$(K \mid f) = \left(\begin{array}{cc|c} 5 & -1 & -4 \\ -1 & 5 & 4 \end{array}\right) \overset{r_2 + \frac{r_1}{5}}{\sim} \left(\begin{array}{cc|c} 5 & -1 & -4 \\ 0 & \frac{24}{5} & \frac{16}{5} \end{array}\right)$$

$$x_2^* = \frac{16}{24} = \frac{2}{3}, \quad 5x_1^* = x_2 - 4 = \frac{2}{3} - 4 = -\frac{10}{3} \Rightarrow x_1^* = -\frac{2}{3}.$$

Thus, the closest point is

$$v^* = x_1^* v_1 + x_2^* v_2 = -\frac{2}{3}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -4/3 \\ 4/3 \end{pmatrix},$$

at a distance of

$$\| b - v^* \| = \sqrt{\left(\tfrac{8}{3}\right)^2 + \left(\tfrac{4}{3}\right)^2 + \left(\tfrac{4}{3}\right)^2} = \frac{1}{3}\sqrt{96} = \frac{4}{3}\sqrt{6} \approx 3.27$$