Workshop 2 - Lecture 13

In these notes, we study examples of "basic" or "fundamental" proofs.

ex Prove the "uniqueness of the additive identity" in a vector space. That is, if V is a vector space & u is an element such that $u+v=v$ for all $v \in V$, then $u=0$.

Pf Suppose that V is a vector space & $u \in V$ is an additive identity. We compute then that

$$0 = u + 0 = u,$$

completing the proof. /

ex Prove uniqueness of additive inverses; for each $v \in V$, the additive inverse is unique.

Pf Let $v \in V$. Suppose $u \in V$ is an additive inverse for v. Then $u+v=0$. Hence, adding $-v$ to both sides,

$$u + v + -v = -v,$$ and so $u = -v$. //

These proofs, once the "right" perspective is adopted, are often quite short.

Most proofs fall somewhere between these extremes. Consider the following.

Prove that the space of continuous functions $f:\mathbb{R}\to\mathbb{R}$ is a vector space. You may use the fact that $\mathcal{F}(\mathbb{R}) = \{$all functions $f:\mathbb{R}\to\mathbb{R}\}$ is a vector space as well as basic rules of limits.

Pf Let $C(\mathbb{R})$ be the space of continuous functions $f:\mathbb{R}\to\mathbb{R}$.
Because $C(\mathbb{R}) \underset{\text{"is contained in"}}{\subseteq} \mathcal{F}(\mathbb{R})$, we may use the subspace criteria in Proposition 2.9. ~~We verify the necessary hypotheses to Prop 2.9~~ Thus, we need to check closure of addition & scalar multiplication.

$\boxed{\text{closure} +}$ Let $f, g \in C(\mathbb{R})$. ~~Then~~ Let $x \in \mathbb{R}$. We see that

$$\lim_{a\to x} (f+g)(a) = \lim_{a\to x} f(a) + g(a) = \lim_{a\to x} f(a) + \lim_{a\to x} g(a).$$

Because f & g are continuous, we have

$$\lim_{a\to x} f(a) + \lim_{a\to x} g(a) = f(x) + g(x) = (f+g)(x).$$

Thus, $f+g$ is continuous because $\lim_{a\to x} (f+g)(a) = (f+g)(x)$ for any $x \in \mathbb{R}$.

$\boxed{\text{Closure} \cdot}$ Let $f \in C(\mathbb{R})$, $c \in \mathbb{R}$. ~~Then~~ Let $x \in \mathbb{R}$. We see that

$$\lim_{a\to x} (cf)(a) = \lim_{a\to x} cf(a) = c \lim_{a\to x} f(a) = cf(x).$$

Thus, cf is continuous because $\lim_{a\to x} (cf)(a) = (cf)(x)$. //

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