Lecture 2 §1.3

Using matrix multiplication, we can shorten the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + &\cdots + a_{1n}x_n = b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + &\cdots + a_{mn}x_n = b_m \end{aligned}$$

to $\quad Ax = b, \quad$ where $A = (a_{ij}),\ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},\ b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

We will use matrix equations from now on.

Gaussian Elimination (GE)

To solve $Ax = b$, we first form the augmented matrix $(A \mid b)$ and then use "LSO 1".

Note that applying LSO 1 is the same as the following:

$\boxed{\text{Elementary Row Operation 1}}$ (ERO 1)

Add a multiple of one row in $(A \mid b)$ to another.

ex (continued from lecture 1)

1st pivot, 2nd pivot, 3rd pivot

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 2 & 5 & -2 & 4 \\ 3 & 4 & -3 & 1 \end{array}\right) \underset{r_3 - 3r_1}{\overset{r_2 - 2r_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & 3 & -7 \end{array}\right) \overset{r_3 + 2r_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right)$$

Back substitution $\rightsquigarrow$ $\boxed{\begin{aligned} x_3 &= -1 \\ x_2 &= 2 \\ x_1 &= -4 \end{aligned}}$

Using GE, we see that this technique works because our "pivots" are always nonzero. Such matrices A are "regular".

Elementary Matrices

Recall A is $m\times n$, solving $Ax=b$.

The **elementary matrix** E associated to an ERO is obtained by doing the ERO to the $m\times m$ identity matrix, I_m.

ex (continued) $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

$$\updownarrow$$

$$r_2 - 2r_1$$

Multiplying A on the left by an elementary matrix is the same as doing the associated ERO to A (or $(A|b)$).

ex (continued)

$$E_1(A|b) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 0 & 1 & 2 & 0 \\ 3 & 4 & -3 & 1 \end{array}\right)$$

$$E_2\cdot(E_1(A|b)) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & 3 & -7 \end{array}\right)$$

$$E_3(E_2E_1(A|b)) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right).$$

Fact The inverse of an elementary matrix of type 1 is given by multiplying the off diagonal coefficient by -1.

ex $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Check $E_1E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$.

We will see a generalization of this later

Note This is related to the mistake I made in class on Monday 9/8