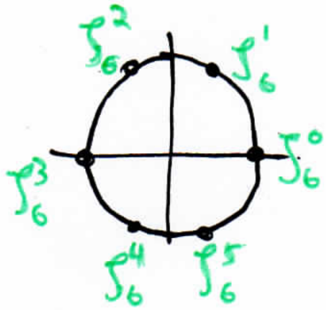
Example: Computing the sampled exponential basis $n=6$

To compute the sampled exponential basis $\omega_0 \ldots \omega_5$, first we compute and label the powers of ζ_6. $\zeta_6^0 = 1$

$$\zeta_6 = e^{\frac{2\pi i}{6}} = e^{\frac{\pi}{3} i} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$



$$\zeta_6^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \qquad \zeta_6^3 = -1 \quad \zeta_6^4 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i \quad \zeta_6^5 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

Then we recall that, because higher powers will start to repeat, this suffices to find all sampled exponentials (e.g., $\zeta_6^1 = \zeta_6^7 = \zeta_6^{13} \ldots$)

$$\omega_0 = (\zeta_6^0, \zeta_6^0, \zeta_6^0, \zeta_6^0, \zeta_6^0, \zeta_6^0) = (1, 1, 1, 1, 1, 1)$$

$$\omega_1 = (\zeta_6^0, \zeta_6^1, \zeta_6^2, \zeta_6^3, \zeta_6^4, \zeta_6^5) = \left(1, \frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2} i, \frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$$

$$\omega_2 = (\zeta_6^0, \zeta_6^2, \zeta_6^4, \zeta_6^6, \zeta_6^8, \zeta_6^{10}) = (\zeta_6^0, \zeta_6^2, \zeta_6^4, \zeta_6^0, \zeta_6^2, \zeta_6^4)$$

$$= \left(1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$$

$$\omega_3 = (\zeta_6^0, \zeta_6^3, \zeta_6^0, \zeta_6^3, \zeta_6^0, \zeta_6^3) = (1, -1, 1, -1, 1, -1)$$

etc....

Example: Computing the k^{th} Fourier Coefficient

Given the sampled values $\vec{f} = (1, 2, 0, -1, 0, 0)$, compute the Fourier coefficients.

Solution Here, we have 6 sample values, so we use the basis computed in the previous example.

$$c_0 = \langle \vec{f}, \omega_0 \rangle = \frac{1}{6}(1\cdot\bar{1} + 2\cdot\bar{1} + 0\cdot\bar{1} + -1\cdot\bar{1} + 0\cdot\bar{1} + 0\cdot\bar{1})$$

$$= \frac{1}{6}(1 + 2 - 1) = \frac{1}{3}$$

$$c_1 = \langle \vec{f}, \omega_1 \rangle = \frac{1}{6}\left(1\cdot\bar{1} + 2\cdot\overline{\frac{1}{2} + \frac{\sqrt{3}}{2}i} + 0 - 1\cdot\overline{-1} + 0 + 0\right)$$

$$= \frac{1}{6}(1 + 1 - \sqrt{3}\,i - 1) = \frac{1}{6} - \frac{\sqrt{3}}{6}i$$

$$c_2 = \langle \vec{f}, \omega_2 \rangle = \frac{1}{6}\left(1\cdot\bar{1} + 2\overline{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)} + 0 - 1\cdot\bar{1} + 0 + 0\right)$$

$$= \frac{1}{6}(1 - 1 - \sqrt{3} + 1) = \frac{1}{6} - \frac{\sqrt{3}}{6}i$$

$$c_3 = \langle \vec{f}, \omega_3 \rangle = \frac{1}{6}(1\cdot 1 + 2\cdot -1 + 0 + -1\cdot -1 + 0 + 0)$$

$$= \frac{1}{6} - \frac{2}{6} + \frac{1}{6} = 0$$

$$c_4 = \frac{1}{6} + \frac{\sqrt{3}}{6}i$$

$$c_5 = \frac{1}{6} + \frac{\sqrt{3}}{6}i$$

Example: Expanding & simplifying the low frequency interpolant

Given the sample values from before, write the real valued low frequency interpolant for $\vec{F}$.

Solution|| The low frequency interpolant for $n=6$ is given by

$$p(x) = c_{-3}e^{-i3x} + c_{-2}e^{-i2x} + c_{-1}e^{-ix} + \underset{\substack{\uparrow \\ e^0 = 1}}{c_0} + c_1 e^{ix} + c_2 e^{i2x}.$$

Recall also that by aliasing, $c_k = c_{k-6}$ (so $c_{-3} = c_3$, $c_{-2} = c_4$, $c_{-1} = c_5$)

So our interpolant is

$$0 + \left(\frac{1}{6} + \frac{\sqrt{3}}{6}i\right)\left(\cos(-2x) + i\sin(-2x)\right) + \left(\frac{1}{6} + \frac{\sqrt{3}}{6}i\right)\left(\cos(-x) + i\sin(-x)\right)$$

$$+ \frac{1}{3} + \left(\frac{1}{6} - \frac{\sqrt{3}}{6}i\right)\left(\cos x + i\sin x\right) + \left(\frac{1}{6} - \frac{\sqrt{3}}{6}i\right)\left(\cos(2x) + i\sin(2x)\right).$$

To simplify, we recall that $\cos(-a) = \cos(a)$ and $\sin(-a) = -\sin(a)$ (cos is even and sin is odd). Thus,

$$p(x) = \frac{1}{6}\cos 2x + \frac{\sqrt{3}}{6}\sin 2x + i\frac{\sqrt{3}}{6}\cos 2x + -\frac{i}{6}\sin 2x$$

$$+ \frac{1}{6}\cos x + \frac{\sqrt{3}}{6}\sin x + i\frac{\sqrt{3}}{6}\cos x - \frac{i}{6}\sin x$$

$$+ \frac{1}{3} + \frac{1}{6}\cos x + \frac{\sqrt{3}}{6}\sin x + -i\frac{\sqrt{3}}{6}\cos x + \frac{i}{6}\sin x$$

$$+ \frac{1}{6}\cos 2x + \frac{\sqrt{3}}{0}\sin 2x - i\frac{\sqrt{3}}{6}\cos x + \frac{i}{6}\sin 2x$$

$$= \frac{1}{3} + \frac{1}{3}\cos x + \frac{\sqrt{3}}{3}\sin x + \frac{1}{3}\cos 2x + \frac{\sqrt{3}}{6}\sin 2x$$

So, our interpolant ends up as real valued without modification (which will not always be the case).

Example: The Fast Fourier Transform

Suppose we are given the sampled vector $f=(1,1,0,-1)$.

We will use the FFT to compute the coefficients c_k.

Solution $n=2^2$, $r=2$. We start by breaking f into f^{even} and f^{odd} recursively, and rebuild the coefficients; recall $c_k = c_k^{even} + e^{\frac{-2\pi i k}{n}} c_k^{odd}$

$r=2$ $\qquad f=(1,1,0,-1)$

$r=1$ $\qquad f^{even}=(1,0) \qquad f^{odd}=(1,-1)$

$r=0$ $\qquad f^{even}=(1) \qquad f^{odd}=(0) \qquad f^{even}=(1) \qquad f^{odd}=(-1)$

At depth 0, $c_0=f_0$.

$r=0$ $\qquad c=(1) \qquad c=(0) \qquad c=(1) \qquad c=(-1)$

$r=1$

$$c=\left(\frac{1+e^{-\pi i\cdot 0}\cdot 0}{2}, \frac{1+e^{-\pi i\cdot 1}\cdot 0}{2}\right) = \left(\frac{1+0}{2}, \frac{1+-1\cdot 0}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$c=\left(\frac{1+e^{-\pi i\cdot 0}\cdot -1}{2}, \frac{1+e^{-\pi i\cdot 1}\cdot -1}{2}\right) = \left(\frac{1+1\cdot -1}{2}, \frac{1+-1\cdot -1}{2}\right) = \left(\frac{1-1}{2}, \frac{1+1}{2}\right) = (0, 1)$$

$r=2$

$$c=\left(\frac{\frac{1}{2}+e^{-\frac{\pi i}{2}\cdot 0}\cdot 0}{2}, \frac{\frac{1}{2}+e^{-\frac{\pi i}{2}\cdot 1}\cdot 1}{2}, \frac{\frac{1}{2}+e^{-\frac{2\pi i}{2}}\cdot 0}{2}, \frac{\frac{1}{2}+e^{-\frac{3\pi i}{2}}\cdot 1}{2}\right)$$

$$=\left(\frac{1}{4}, \frac{1}{4}+\frac{-i\cdot 1}{2}, \frac{1}{4}, \frac{1}{4}+\frac{i\cdot 1}{2}\right) = \left(\frac{1}{4}, \frac{1}{4}-\frac{i}{2}, \frac{1}{4}, \frac{1}{4}+\frac{i}{2}\right).$$